Problem 10.1

We still have: $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$
but this time, the boundary conditions are:

$$\psi(-a/2) = \psi(a/2) = \emptyset.$$

so start w/ the general solution:

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad \text{w/ } k \equiv \sqrt{\frac{2mE}{\hbar^2}}$$

& impose the left boundary: $\psi(-a/2) = Ae^{-iak/2} + Be^{iak/2} = \emptyset$
gives

$$B = -Ae^{-iak}, \text{ \& then}$$

$$\psi(x) = A\left[e^{ikx} - e^{-ik(x+a)}\right]$$

Then the right-hand boundary condition is:

$$\psi(a/2) = A\left[e^{ika/2} - e^{-ik(3a/2)}\right]$$

& for this to be zero, we need $e^{ika/2} = e^{-ik3a/2} \Rightarrow e^{ik\cdot 2a} = 1$

This gives us $k2a = n\cdot 2\pi$ or $\boxed{k = \frac{n\pi}{a}}$ for integer n

Now, we can reduce $\psi(x)$ to trigonometric functions:

$$\psi(x) = A\left[e^{i\frac{n\pi x}{a}} - e^{-i\frac{n\pi}{a}(x+a)}\right] = A\left[e^{i\frac{n\pi x}{a}} - \underbrace{e^{-in\pi}}_{=(-1)^n} \cdot e^{-i\frac{n\pi x}{a}}\right]$$

For n odd, then, we have: $\psi_n(x) = A\left[e^{in\pi x/a} + e^{-in\pi x/a}\right] \sim \cos\left(\frac{n\pi x}{a}\right)$
while for n even,

$$\psi_n(x) = A\left[e^{i\frac{n\pi x}{a}} - e^{-in\pi x/a}\right] \sim \sin\left(\frac{n\pi x}{a}\right)$$

The normalization, in either case, is the usual $\sqrt{\frac{2}{a}}$, & we have:

$$\boxed{\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right) & n \text{ odd} \\ \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) & n \text{ even} \end{cases}}$$

in either case (n even or odd) we have the spectrum: $k = \frac{n\pi}{a} \Rightarrow \sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{a}$,

$$\Updownarrow$$

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2ma}}$$

(Note that you can also take the $\emptyset \to a$ well solutions & shift: $x \to x - a/2$ & use trig. identities to get these quickly).

Problem 10.2

After the measurement of $E = \frac{\pi^2 \hbar^2}{2ma^2}$, we know the particle is in the $n=1$ state of the infinite square well. It remains in the state:

$$\psi_1(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right) & -a/2 \le x \le a/2 \\ \emptyset & \text{else} \end{cases} \qquad (*)$$

since this is a stationary state. When we turn on the harmonic oscillator potential, (*) represents our initial state: $\Psi(x,\emptyset)$, & we know that a measurement of $E = \frac{1}{2}\hbar\omega$ corresponds to the ground state of the harmonic oscillator, & will be measured w/ prob. given by $A_0^* A_0$ in the decomposition:

$$\Psi(x,t) = \sum_{j=0}^{\infty} A_j \underbrace{\psi_j(x)}_{\psi_j(x) \text{ is } j^{\text{th}} \text{ harmonic oscillator stationary state, \& } E_j = (j+\frac{1}{2})\hbar\omega \text{ is the associated energy.}} e^{-iE_j t/\hbar}$$

where these coefficients are set by the requirement:

$$\Psi(x,\emptyset) = \sum_{j=0}^{\infty} A_j \psi_j(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right) & -a/2 \le x \le a/2 \\ \emptyset & \text{else.} \end{cases} \equiv \Psi(x)$$

so we need to compute: $A_0 = \int_{-\infty}^{+\infty} \psi_0(x) \Psi(x)\, dx$

$$= \boxed{\int_{-a/2}^{+a/2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right) dx}$$

& then $P(\emptyset) = A_0^* A_0$.