The Infinite Square Well II

Lecture 7

Physics 342
Quantum Mechanics I

Monday, February 8th, 2010

We will review some general properties of stationary states in quantum mechanics using the infinite square well solution as our vehicle. In particular, we will discuss the role of the special solutions to Schrödinger's equation:

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}}\,\sin\!\left(\frac{n\,\pi\,x}{a}\right)\,e^{-i\,\frac{n^2\,\pi^2\,\hbar}{2\,m\,a^2}\,t}. \tag{7.1}$$

These have, as solutions, the following interesting properties:

1. They have densities that are time-independent, $\Psi_n^*\,\Psi_n = \rho(x)$ a function of x only (stationary).

2. They have definite energy. The Hamiltonian, $H(x,p) = \frac{p^2}{2\,m} + V(x)$ is, numerically, the total energy of a particle of mass m in a potential $V(x)$. We can compute the quantum mechanical expectation value by considering:

$$H\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)\,\Psi_n(x,t) = \left(-\frac{\hbar^2}{2\,m}\frac{\partial^2}{\partial x^2} + V(x)\right)\,\Psi_n(x,t) = E_n\,\Psi_n(x,t) \tag{7.2}$$

where $E_n = \frac{n^2\,\pi^2\,\hbar^2}{2\,m\,a^2}$ is the quantum mechanical energy of a particle in state $\Psi_n(x,t)$. This result follows as a direct consequence of the time-independent Schrödinger equation. Then the expectation value of H, what we would call the average energy, is:

$$\begin{aligned} \langle H \rangle &= \int_0^a \Psi_n^*(x,t)\,H\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)\,\Psi_n(x,t)dx \\ &= E_n\,\int_0^a \Psi_n(x,t)^*\,\Psi_n(x,t)\,dx \\ &= E_n \end{aligned} \tag{7.3}$$

and we can compute the expectation value of H^2 in a similar manner:

$$\begin{aligned}
\langle H^2 \rangle &= \int_0^a \Psi_n^*(x,t)\, H\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)\, H\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)\, \Psi_n(x,t) dx \\
&= E_n \int_0^a \Psi_n^*(x,t)\, H\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)\, \Psi_n(x,t)\, dx \\
&= E_n^2 \int_0^a \Psi_n^*(x,t)\, \Psi_n(x,t)\, dx = E_n^2,
\end{aligned} \tag{7.4}$$

so that $\sigma^2 = \langle H^2 \rangle - \langle H \rangle^2 = 0$, i.e. there is no spread in the energy measurement.

Using these "basis" functions, we can add solutions with arbitrary coefficients, so that a general solution to Schrödinger's equation for the infinite square well can be written as:

$$\Psi(x,t) = \sum_{n=1}^{\infty} A_j\, \Psi_n(x,t). \tag{7.5}$$

The coefficients $\{A_j\}_{j=1}^{\infty}$ are set by the provided initial function $\Psi_0(x)$ by demanding that:

$$\Psi(x,0) = \sum_{n=1}^{\infty} A_j \sqrt{\frac{2}{a}}\, \sin\left(\frac{n\,\pi\,x}{a}\right) = \Psi_0(x). \tag{7.6}$$

The normalization requirement becomes:

$$\sum_{j=1}^{\infty} A_j^*\, A_j = 1. \tag{7.7}$$

This result is meant to remind you of the ball-dropping experiment we did to introduce probabilities $P(j)$ – in (7.7), the expression $A_j^*\, A_j$ is playing the role of $P(j)$. The coefficient A_j itself can be interpreted as "the amount of $\Psi_j(x,t)$ in $\Psi(x,t)$."

If we think of the average energy of our general $\Psi(x,t)$, then we could write:

$$\begin{aligned}
\langle H \rangle &= \int_0^a \Psi^*(x,t)\, H\!\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)\, \Psi(x,t)\, dx \\
&= \int_0^a \Psi^*(x,t) \sum_{n=1}^{\infty} A_n\, H\!\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)\, \Psi_n(x,t) \\
&= \int_0^a \Psi^*(x,t) \sum_{n=1}^{\infty} A_n\, E_n\, \Psi_n(x,t) \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} A_k^*\, A_n\, E_n\, \left[\int_0^a \Psi_k^*(x,t)\, \Psi_n(x,t)\, dx\right] \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} A_k^*\, A_n\, E_n\, [\delta_{kn}] \\
&= \sum_{n=1}^{\infty} E_n\, (A_n^*\, A_n)\,,
\end{aligned} \tag{7.8}$$

using the orthonormality of $\Psi_n(x,t)$. The last line should remind you of the averages we took, again for the ball-dropping experiment, that were of the form:

$$\langle f(j) \rangle = \sum_{j=1}^{\infty} f(j)\, P(j). \tag{7.9}$$

Evidently, the average energy for a general state of the infinite square well can be interpreted as the average measurement of the energy where the *only* available experimental outcomes are the discrete set $\{E_n\}_{n=1}^{\infty}$. To this observation, we add the measurement "assumption": If you measure the energy of a particle that has $\Psi(x,t)$, you will obtain the value E_n with probability $A_n^*\, A_n$ and immediately after this measurement, the particle will be in the state $\Psi_n(x,t)$. That is, somehow, you have a general state $\Psi(x,t)$, and after measurement, you *know* the state is $\Psi_n(x,t)$. Furthermore, as mentioned above, once the particle is in the state $\Psi_n(x,t)$, *it stays there.*

While we will develop and discuss these ideas specifically for the infinite square well, you should think about just how general they might be ...

Homework

Reading: Griffiths, pp. 30–40.

Problem 7.1

Griffiths 2.3. No solutions for negative energy in the infinite square well.

Problem 7.2

Griffiths 2.5. Time dependence comes from mixing stationary states.

Problem 7.3

Griffiths 2.39. The “revival” time for the infinite square well.