# Angular Momentum Addition II

Lecture 30

Physics 342
Quantum Mechanics I

Wednesday, April 14th, 2010

Our goal is to take eigenstates of the $z$-component and magnitude squared of two separate particle spin operators: $|s_1\, m_1\rangle$ and $|s_2\, m_2\rangle$ with:

$$\begin{array}{ll} {}^1S_z\,|s_1\,m_1\rangle = \hbar\, m_1 |s_1\,m_1\rangle & {}^1S^2\,|s_1\,m_1\rangle = \hbar^2\, s_1\,(s_1+1)\,|s_1\,m_1\rangle \\ {}^2S_z\,|s_2\,m_2\rangle = \hbar\, m_2 |s_2\,m_2\rangle & {}^2S^2\,|s_2\,m_2\rangle = \hbar^2\, s_2\,(s_2+1)\,|s_2\,m_2\rangle, \end{array} \tag{30.1}$$

and form linear combinations of the product (two-particle) states

$$|s_1\,m_1\rangle\,|s_2\,m_2\rangle \tag{30.2}$$

that are eigenstates of $S_z$ and $S^2$, associated with the total spin

$$\mathbf{S} = {}^1\mathbf{S} + {}^2\mathbf{S}. \tag{30.3}$$

Call those eigenstates $|s\,m\rangle$, so

$$S_z\,|s\,m\rangle = \hbar\, m\,|s\,m\rangle \qquad S^2\,|s\,m\rangle = \hbar^2\, s\,(s+1)\,|s\,m\rangle. \tag{30.4}$$

We want the coefficients that give the decomposition of $|s\,m\rangle$ in terms of $|s_1\,m_1\rangle\,|s_2\,m_2\rangle$. For fixed $s_1$ and $s_2$ (typical), we want the coefficients below:

$$|s\,m\rangle = \sum_{m_j+m_k=m} C^{s_1\,s_2\,s}_{m_1\,m_2\,m}\,|s_1\,m_1\rangle\,|s_2\,m_2\rangle. \tag{30.5}$$

We know that $s = s_1 + s_2 \longrightarrow |s_1 - s_2|$ in integer steps, and for each value of $s$, $-s \le m \le s$. Then we automatically know one eigenstate:

$$|s_1+s_2 \quad s_1+s_2\rangle = |s_1 \quad s_1\rangle\,|s_2 \quad s_2\rangle. \tag{30.6}$$

Using the lowering operator $S_- \equiv {}^1S_- + {}^2S_-$, we can act on both sides of this equality to find the $|s_1+s_2 \quad s_1+s_2-1\rangle$ state, and continue all the way down to $|s_1+s_2 \quad -(s_1+s_2)\rangle$.

But then we have to find the next series in $s$, i.e. $s = s_1 + s_2 - 1$. If we knew the state $|s_1 + s_2 - 1 \;\; s_1 + s_2 - 1\rangle$, we could again apply the lowering operator to find all the states with $s = s_1 + s_2 - 1$. To start the process off, we use the fact that we know $|s_1 + s_2 \;\; s_1 + s_2 - 1\rangle$ is orthogonal to $|s_1 + s_2 - 1 \;\; s_1 + s_2 - 1\rangle$, i.e.:

$$\langle s_1 + s_2 \;\; s_1 + s_2 - 1 | s_1 + s_2 - 1 \;\; s_1 + s_2 - 1\rangle = 0, \tag{30.7}$$

and we can use this for some initial combination, say

$$\alpha\,|s_1 \;\; s_1 - 1\rangle\,|s_2 \;\; s_2\rangle + \beta|s_1 \;\; s_1\rangle\,|s_2 \;\; s_2 - 1\rangle \tag{30.8}$$

to relate $\alpha$ and $\beta$, with normalization providing a second equation to fix the value of the pair $\alpha$, $\beta$.

As a final note, the orthogonality of the two-particle states is expressed via:

$$\langle s_1\, m_1|\,\langle s_2\, m_2|\,\big(|s_1\, m_1'\rangle\,|s_2\, m_2'\rangle\big) = \langle s_1\, m_1|s_1\, m_1'\rangle\,\langle s_2\, m_2|s_2\, m_2'\rangle = \delta_{m_1\, m_1'}\,\delta_{m_2\, m_2'} \tag{30.9}$$

(given two different $m_1$, $m_1'$ and $m_2$, $m_2'$) so that bras for particle one interact only with kets for particle one, and the same for particle 2 (i.e. again, the operators and states for particles one and two do not talk to each other).

Finally, referring back to (30.5), we can look up the coefficients associated with the decomposition, these are the Clebsch-Gordon coefficients. An example of a table of relevant values for $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$ is shown in Figure 30.1.

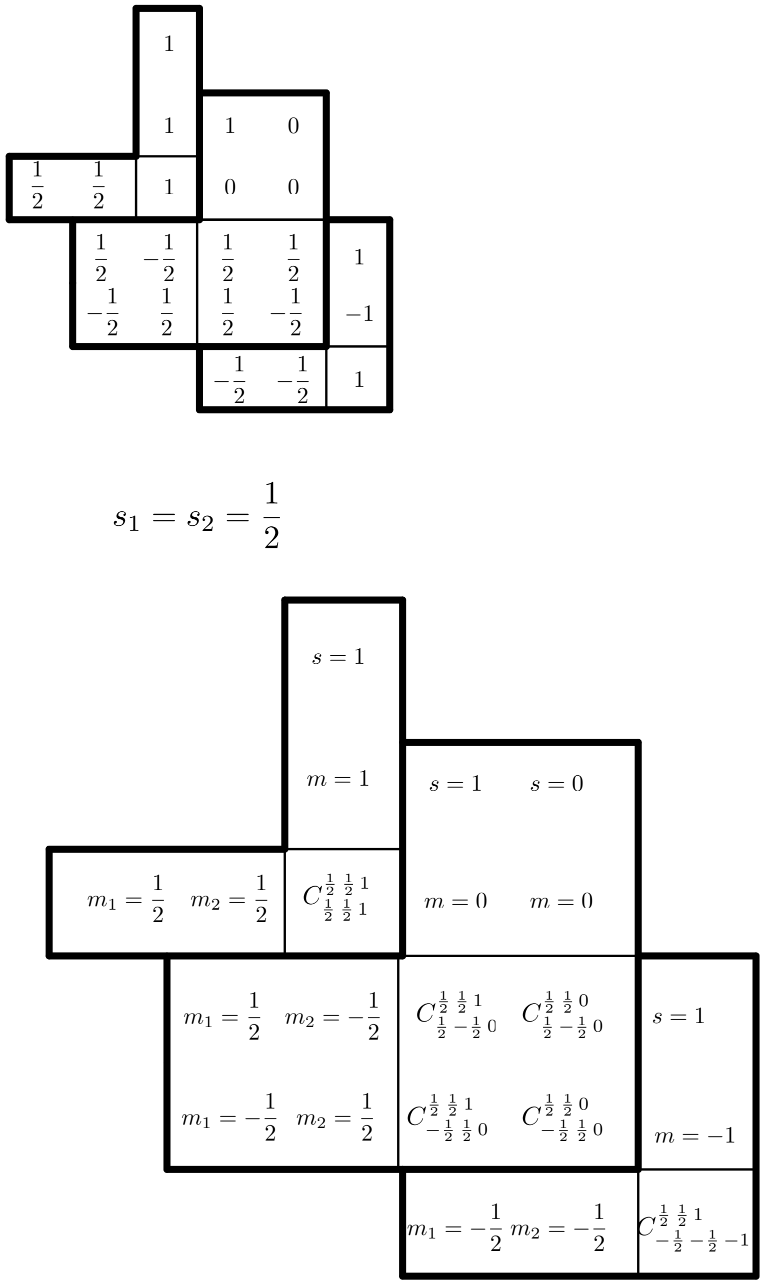


Figure 30.1: The Clebsch-Gordon coefficients for $s_1 = s_2 = \frac{1}{2}$ – on top, the table copied from Table 4.8 (Griffiths), and below, the explicit meaning of each entry. Note that the values in the numerical table are really the coefficients $C^{s_1\,s_2\,s}_{m_1\,m_2\,m}$ squared (with minus signs indicating subtraction).

## Homework

Reading: Griffiths, pp. 185–200.

### Problem 30.1

Form the eigenstates of $J^2$ for $\mathbf{J} = \mathbf{L} + \mathbf{S}$ with $\ell = 1$, $s = \frac{1}{2}$ (think of an electron in some $\ell = 1$ state of Hydrogen, while ignoring the proton spin) – you will get four states of angular momentum $\frac{3}{2}$ and two states of angular momentum $\frac{1}{2}$ (that's $j = 1 + \frac{1}{2}$ and $j = 1 - \frac{1}{2}$). Use the ladder approach to explicitly construct all states by finding the "top" state and working down to the bottom. Check your decompositions using the Clebsch-Gordon table (Table 4.8 on p. 188).

### Problem 30.2

Griffiths 4.35. Practice with total spin states.