Addition of Angular Momentum

Lecture 28

Physics 342
Quantum Mechanics I

Wednesday, April 7th, 2010

We consider the physical effects of spin, and for this we need an interaction that couples to spin. A uniform, or weakly non-uniform magnetic field can be used to calculate approximate solutions to Schrödinger's equation that involve the spin of the electron (say).

With spin in place, we need to explore composite angular momentum systems. For example, in Hydrogen, we have orbital angular momentum (associated with the quantum numbers ℓ and m) in addition to the intrinsic spin of the electron. What, then, are the allowed values of total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$?

28.1   Spin and the Magnetic Field

As noted at the end of last time, there is a relation between microscopic, point particle spin and a more macroscopic picture (the story we tell ourselves). The most important connection is in the interaction with an external magnetic field. This provides a Hamiltonian, and therefore definite energies associated with the spin up and spin down configurations.

The spin of the electron allows us to think of an electron's intrinsic magnetic dipole moment. We have charge e spread out through a sphere (of zero radius) that is spinning with some angular velocity. Then the usual Hamiltonian for a dipole moment in a magnetic field holds;

$$H = -\gamma \, \mathbf{S} \cdot \mathbf{B} \tag{28.1}$$

where γ is the "gyromagnetic ratio" for the electron (since the electron is a point particle, we have no clean way, classically, to calculate its dipole

moment, so we simply assume the dipole is proportional to the angular velocity and depends on the amount of charge carried by the electron) – this turns out to be $\gamma = -\frac{e}{m}$. Regardless, suppose we take a homogenous magnetic field pointing in the $\hat{\mathbf{z}}$ direction: $\mathbf{B} = B_0 \hat{\mathbf{z}}$, then the Hamiltonian is $H = -\gamma B_0 S_z$, and we can write this in matrix form:

$$H \doteq -\frac{\gamma B_0 \hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{28.2}$$

This Hamiltonian clearly commutes with S_z (since it is, itself, proportional to S_z and $[S_z, S_z] = 0$) and S^2 (since $[S_z, S^2] = 0$). Then the eigenstates of the Hamiltonian are eigenstates of S_z, i.e. χ_+ and χ_-, and the eigenvalues (allowed energies) can be determined via

$$H \chi_+ = -\frac{\gamma B_0 \hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{2} B_0 \gamma \hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{28.3}$$

so that $E_\pm = \mp\frac{1}{2} B_0 \gamma \hbar$. The lower eigenvalue is associated with spin up, or parallel alignment (true also for classical magnetic dipole orientations). Then a generic initial state

$$\chi(t = 0) = \cos(\alpha) \chi_+ + \sin(\alpha) \chi_- \tag{28.4}$$

(where we are ensuring that $\chi^\dagger \chi = 1$ by using a trigonometric one-parameter family defined by α) has time-dependence given by Schrödinger's equation: $i \hbar \frac{\partial}{\partial t} \chi(t) = H \chi(t)$:

$$\chi(t) = \cos(\alpha) e^{-\frac{i E_+ t}{\hbar}} \chi_+ + \sin(\alpha) e^{-\frac{i E_- t}{\hbar}} \chi_-. \tag{28.5}$$

The expectation values are given by $\langle \mathbf{S} \rangle$, just matrix multiplication here – take $\langle S_x \rangle$ first,

$$\langle S_x \rangle = \begin{pmatrix} \cos(\alpha) e^{\frac{i E_+ t}{\hbar}} & \sin(\alpha) e^{\frac{i E_- t}{\hbar}} \end{pmatrix} \left(\frac{\hbar}{2} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\alpha) e^{-\frac{i E_+ t}{\hbar}} \\ +\sin(\alpha) e^{-\frac{i E_- t}{\hbar}} \end{pmatrix}$$

$$= \hbar \cos\alpha \sin\alpha \cos(B_0 \gamma t).$$
$$\tag{28.6}$$

By identical calculation for S_y and S_z, we obtain:

$$\langle S_y \rangle = -\hbar \cos\alpha \sin\alpha \sin(B_0 \gamma t) \quad \langle S_z \rangle = \frac{1}{2}\hbar \cos(2\alpha). \tag{28.7}$$

If we define $\beta \equiv 2\alpha$, then $\sin\alpha \cos\alpha = \frac{1}{2}\sin\beta$, and our expectation values are

$$\langle S_x \rangle = \frac{1}{2}\,\hbar\,\sin(\beta)\,\cos(B_0\,\gamma\,t) \quad \langle S_y \rangle = \frac{1}{2}\,\hbar\,\sin(\beta)\,\sin(B_0\,\gamma\,t) \quad \langle S_z \rangle = \frac{1}{2}\,\hbar\,\cos(\beta).$$
(28.8)

These equations define a vector that points towards a circle of radius $\frac{1}{2}\,\hbar\,\sin\beta$ and goes around the circle with frequency $\omega = B_0\,\gamma$, the Larmor frequency.

28.2 Stern-Gerlach Experiment

Since spin interacts with the magnetic field just as classical dipole moments do, we expect the usual sort of predictions from electrodynamics to hold – in particular, for a dipole in a magnetic field, there is a force $\mathbf{F} = \nabla\,(\gamma\,\mathbf{S}\cdot\mathbf{B})$. Suppose we construct a magnetic field that points in the $\hat{\mathbf{z}}$ direction, and has magnitude $\mathbf{B} = (B_0 + \alpha z)\,\hat{\mathbf{z}}$ – note that this cannot be the full magnetic field since $\nabla\cdot\mathbf{B} = \alpha \neq 0$, but we can ignore the other components since our Hamiltonian is going to be determined, in this case, by S_z, that's where the physics plays a role.

We imagine a packet of electrons (attached to atoms, for example), moving through a field region. From the point of view of the electrons, we have the following *time-dependent* Hamiltonian:

$$H = \begin{cases} 0 & t < 0 \text{ and } t > T \\ -\gamma(B_0 + \alpha\,z)\,S_z & 0 < t < T \end{cases},$$
(28.9)

so that the electrons are at rest in a region with no field ($t < 0$), then they are subjected to a field ($0 < t < T$), and finally, the field is turned off ($T > 0$) . When in the field, the up and down spins take on two different energies,

$$E_\pm = \mp\frac{1}{2}\,\hbar\,\gamma\,(B_0 + \alpha\,z).$$
(28.10)

At time $t = 0$, the electrons are in some initial spin state, $\chi(t = 0) = \alpha\,\chi_+ + \beta\,\chi_-$. Once inside the field, the spin state evolves in time according to the usual

$$\chi(t) = A\,e^{-\frac{i\,E_+\,t}{\hbar}}\,\chi_+ + B\,e^{-\frac{i\,E_-\,t}{\hbar}}\,\chi_-$$
(28.11)

and here, there is position dependence coming from the energies themselves[1]. When the field is then turned off:

$$\chi(t) = A\,e^{\frac{i\,\gamma\,B_0\,T}{2}}\,e^{i\,\frac{\alpha\,\gamma\,z}{2}\,T}\,\chi_+ + B\,e^{-\frac{i\,\gamma\,B_0\,T}{2}}\,e^{-i\,\frac{\alpha\,\gamma\,z}{2}\,T}\,\chi_-. \qquad (28.12)$$

Now, we are thinking of a Gaussian wavepacket representing the electron, technically, since for $t < 0$ and $t > T$, the zero-potential Hamiltonian acts, and the solutions there are plane waves. But we know that a wavepacket with the phase factor $e^{\pm i\,\frac{\alpha\,\gamma\,z}{2}\,T}$ carries momentum in the z direction given by $\langle p_z \rangle = \pm\frac{\alpha\,\gamma\,T\,\hbar}{2}$ – again, we need to think of the spin sector of the problem as combined with "the rest" of the wavefunction, which we have not specified explicitly. The point is, the beam will split into two beams, in this case (where we are imagining a spin one half particle), and we will get two spots on a screen behind the magnetic field, one where the spin up components go, one for the spin-down components.

This experiment shows, decisively, the quantization of spin, and was an early indicator of the quantum mechanical point of view. It gets better – if we take the upper beam, corresponding to the (z) spin up electrons, and measure the x-component of the spin, we will again split the beam in two, there are two components to S_x (up and down). That's no surprise – but if we now take the x-spin-up beam and measure the component of spin in the z direction, we will get . . . two beams. This is an experimental verification of the uncertainty principle. We took only the original z-spin-up beam, so you would expect to get no z-spin-up the second time around, but the measurement of the x-spins somehow re-introduced z-spin-up (and down) components.

28.3   Addition of Angular Momentum

Classically, angular momenta add, so we can talk about the total angular momentum of, for example, a spinning, orbiting body as the sum of the spin and orbital angular momentum vectors. The same is true for quantum mechanical angular momentum. For Hydrogen, we can have electrons that are in an orbital state $\ell = 0, 1, 2, \ldots$ and, of course, carry there usual intrinsic

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[1]You should be wary of such an expression – we have cobbled together a state from a position-dependent free-particle solution and a time-dependent spinor, using an expression for "energy" that is spatially dependent. Does $\chi(t)$ in (28.11) satisfy Schrödinger's equation, the only metric for a successful wavefunction?

spin $s = \frac{1}{2}$ – we would say the total angular momentum vector *operator* is $\mathbf{J} = \mathbf{L} + \mathbf{S}$.

Of course, we need to go back one step, since in Hydrogen, the electron is not the only particle with spin. We have been ignoring the nucleus, with its one proton, on the grounds that Hydrogen is really a one-body problem – the proton is heavy enough that we can consider its position fixed compared to the electron. The proton's spin, however, is one half ($s = \frac{1}{2}$), just like the electron's, and so to study the total angular momentum of Hydrogen, we must first know how to calculate the total spin angular momentum of the combined electron-proton system. Take Hydrogen in its $\ell = 0$ ground state, so there is no orbital angular momentum to confuse the issue.

The proton and electron each have spin $\frac{1}{2}$, so what is the total spin of Hydrogen? Well the vector operator is

$$\mathbf{S} = \mathbf{S}^1 + \mathbf{S}^2, \tag{28.13}$$

as advertised, just the sum of the operators for particles one and two (the electron and proton in this case, but we are actually in the process of generating more general addition rules). Now, the $\mathbf{S}^1$ operator acts only on the 1 particle, so these two operators don't actually talk to one another. We can form the raising and lowering operators, $S_\pm$ in the obvious way

$$S_\pm = (S_x^1 \pm i\, S_y^1) + (S_x^2 \pm i\, S_y^2) \tag{28.14}$$

and similarly for the rest of the operators of interest: S^2, S_z, etc. Now suppose we have a state in which both particles are "up" w.r.t. their individual S_z operators – what is the total spin of this state, and what is its composite S_z angular momentum? Let's write $|s_1\, m_1\rangle\, |s_2\, m_2\rangle$ to denote a state with particle one in $|s_1\, m_1\rangle$ and particle two in $|s_2\, m_2\rangle$, then

$$S_z \left(\left| \frac{1}{2}\, \frac{1}{2} \right\rangle \left| \frac{1}{2}\, \frac{1}{2} \right\rangle \right) = \left(S_z^1 \left| \frac{1}{2}\, \frac{1}{2} \right\rangle \right) \left| \frac{1}{2}\, \frac{1}{2} \right\rangle + \left| \frac{1}{2}\, \frac{1}{2} \right\rangle S_z^2 \left| \frac{1}{2}\, \frac{1}{2} \right\rangle$$

$$= \hbar\, m_1 \left| \frac{1}{2}\, \frac{1}{2} \right\rangle \left| \frac{1}{2}\, \frac{1}{2} \right\rangle + \hbar\, m_2 \left| \frac{1}{2}\, \frac{1}{2} \right\rangle \left| \frac{1}{2}\, \frac{1}{2} \right\rangle \tag{28.15}$$

$$= \hbar \left| \frac{1}{2}\, \frac{1}{2} \right\rangle \left| \frac{1}{2}\, \frac{1}{2} \right\rangle.$$

Already, knowing only the z-component, we know that this state, made up of two particles, *cannot* be spin $\frac{1}{2}$ – it could be spin 1, 3/2 or higher, but we cannot have a spin $\frac{1}{2}$ particle with z-component of spin angular momentum larger than $s = \frac{1}{2}$.

To find out what s for the system is, we can use S^2. It's easy to construct, although it looks bad:

$$S^2 = \left(S^1\right)^2 + \left(S^2\right)^2 + \mathbf{S}^1 \cdot \mathbf{S}^2 + \mathbf{S}^2 \cdot \mathbf{S}^1. \tag{28.16}$$

The dot product is tricky – it involves components S_x^1, for example, for which we have an explicit representation for $s = \frac{1}{2}$, and so we can calculate them in that context, but what about the general case? Well, we know how the raising and lowering operators work on generic states, so let's write S_x^1 and S_y^2 in terms of these:

$$S_x^i = \frac{1}{2}\left(S_+^i + S_-^i\right) \quad S_y^i = \frac{1}{2}i\left(S_-^i - S_+^i\right) \quad i = 1, 2. \tag{28.17}$$

Then the dot products are:

$$\begin{aligned}
\mathbf{S}^1 \cdot \mathbf{S}^2 &= \frac{1}{4}\left[\left(S_+^1 + S_-^1\right)\left(S_+^2 + S_-^2\right) - \left(S_-^1 - S_+^1\right)\left(S_-^2 - S_+^2\right)\right] + S_z^1 S_z^2 \\
&= \frac{1}{2}\left(S_+^1 S_-^2 + S_-^1 S_+^2\right) + S_z^1 S_z^2
\end{aligned} \tag{28.18}$$

and this is the same for the $\mathbf{S}^2 \cdot \mathbf{S}^1$ product (these operators commute, since they act on different particles). Then we have

$$\begin{aligned}
\mathbf{S}^1 \cdot \mathbf{S}^2 \left|s_1\, m_1\right\rangle \left|s_2\, m_2\right\rangle = {}& \frac{1}{2}\hbar^2 \sqrt{s_1\left(s_1 + 1\right) - m_1\left(m_1 + 1\right)}\sqrt{s_2\left(s_2 + 1\right) - m_2\left(m_2 - 1\right)}\times \\
& \left|s_1\, m_1 + 1\right\rangle \left|s_2\, m_2 - 1\right\rangle \\
& + \frac{1}{2}\hbar^2 \sqrt{s_1\left(s_1 + 1\right) - m_1\left(m_1 - 1\right)}\sqrt{s_2\left(s_2 + 1\right) - m_2\left(m_2 + 1\right)}\times \\
& \left|s_1\, m_1 - 1\right\rangle \left|k_2\, m_2 + 1\right\rangle \\
& + \hbar^2 m_1 m_2 \left|s_1\, m_1\right\rangle \left|s_2\, m_2\right\rangle.
\end{aligned} \tag{28.19}$$

We see the beginnings of a potential "problem" – it is not clear that just any old pair of spin states, though they are eigenstates of their individual operators $\left(S^1\right)^2$ and $\left(S^2\right)^2$, will be eigenstates of the composite operator S^2. As you can also, pretty clearly see, this won't matter for our chosen state, but it induces the question: When are a pair of states in a definite spin state? And related: How can we construct states that are composite eigenstates?

Let's return to our system, which, since it is a "top" state (w.r.t. m_1 and m_2) has a much simpler product:

$$\mathbf{S}^1 \cdot \mathbf{S}^2 \left|\frac{1}{2}\,\frac{1}{2}\right\rangle \left|\frac{1}{2}\,\frac{1}{2}\right\rangle = \hbar^2 \frac{1}{4}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle \left|\frac{1}{2}\,\frac{1}{2}\right\rangle \tag{28.20}$$

(this is because $S_+^1 \left| \frac{1}{2} \frac{1}{2} \right\rangle = 0$ and similarly for $S_+^2 \left| \frac{1}{2} \frac{1}{2} \right\rangle$). The total S^2 is

$$
S^2 \left| s_1 \, m_1 \right\rangle \left| s_2 \, m_2 \right\rangle = \hbar^2 \left(\frac{1}{2} \left(\frac{1}{2} + 1 \right) + \frac{1}{2} \left(\frac{1}{2} + 1 \right) \right) \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle + \hbar^2 \frac{1}{2} \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle
$$

$$
= \hbar^2 \, 2 \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle ,
$$

(28.21)

from which we conclude $s = 1$. We may then write the relation:

$$
\left| 11 \right\rangle = \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle ,
$$

(28.22)

the combination of the two states has yielded a state with spin $s = 1$ and z-component of angular momentum $m = 1$.

But now we are in good shape – we know how to lower the z-component of angular momentum for a state like this, just apply the lowering operator:

$$
S_- \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle = \left(S_-^1 \left| \frac{1}{2} \frac{1}{2} \right\rangle \right) \left| \frac{1}{2} \frac{1}{2} \right\rangle + \left| \frac{1}{2} \frac{1}{2} \right\rangle \left(S_-^2 \left| \frac{1}{2} \frac{1}{2} \right\rangle \right)
$$

$$
= \hbar \sqrt{\frac{1}{2} \left(\frac{1}{2} + 1 \right) - \frac{1}{2} \left(\frac{1}{2} - 1 \right)} \left(\left| \frac{1}{2} \frac{1}{2} - 1 \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle + \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} - 1 \right\rangle \right)
$$

$$
= \hbar \left(\left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle + \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle \right) .
$$

(28.23)

This state has $m = 0$ and S^2 acting on it gives us $s = 1$ again, so it is the $\left| 1 \, 0 \right\rangle$ state, and represents an equal admixture of "up-down" and "down-up" for the two particles. We can normalize it by introducing a factor of $\frac{1}{\sqrt{2} \, \hbar}$, but that's not too important. Hitting the above with S_- again yields the $m = -1$ state, $\left| 1 \, 0 \right\rangle \sim \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle$, and that exhausts the spin 1 contributions.

The set of combinations that yield overall $s = 1$ states is called, for obvious reasons, the "triplet", and we can write out the fully normalized states as follows:

$$
\left| 1 \, 1 \right\rangle = \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle
$$

$$
\left| 1 \, 0 \right\rangle = \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} \frac{1}{2} \right\rangle + \left| \frac{1}{2} \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle \right)
$$

(28.24)

$$
\left| 1 - 1 \right\rangle = \left| \frac{1}{2} - \frac{1}{2} \right\rangle \left| \frac{1}{2} - \frac{1}{2} \right\rangle .
$$

There is one more state of interest, and it comes from considering the "orthogonal" (to the above) linear combination of up-down states:

$$|0\,0\rangle = \frac{1}{\sqrt{2}} \left(\left| \frac{1}{2} \ -\frac{1}{2} \right\rangle \left| \frac{1}{2}\ \frac{1}{2} \right\rangle - \left| \frac{1}{2}\ \frac{1}{2} \right\rangle \left| \frac{1}{2}\ -\frac{1}{2} \right\rangle \right). \tag{28.25}$$

This is a state whose components both have $m = 0$, but it is a total eigenstate $s = 0$ of S^2. That's not entirely obvious, but if you think about the dot-product terms from above, it is pretty easy to verify.

So we see that in this case, the combination of two particles, each of spin $\frac{1}{2}$ yielded four different states, with two different total spin angular momenta ($s = 1, 0$). The question becomes: How, by linear combination of composite states, is it possible to generate eigenstates of S^2? Suppose, for example, that we have two spin 1 particles. Is there a combination that gives, for example, $|1\,0\rangle$ in there (i.e. $s = 1$, $m = 0$)? First of all, note that all we need is the $|1\,1\rangle$ case, then we can apply the lowering operator. Let's think of making a sum of states $|1\,m_1\rangle\,|1\,m_2\rangle$, and pick m_1, m_2 and the coefficients in the sum so as to construct an $s = 1$ eigenstate of S^2. From (28.19), we know that there will be combinations in the S^2 operator that look like:

$$\mathbf{S}^1 \cdot \mathbf{S}^2\, |1\,m_1\rangle\,|1\,m_2\rangle = \frac{1}{2}\,\hbar^2\,\sqrt{2 - m_1\,(m_1 + 1)}\,\sqrt{2 - m_2\,(m_2 - 1)}\,|1\,m_1 + 1\rangle\,|1\,m_2 - 1\rangle$$
$$+ \frac{1}{2}\,\hbar^2\,\sqrt{2 - m_1\,(m_1 - 1)}\,\sqrt{2 - m_2\,(m_2 + 1)}\,|1\,m_1 - 1\rangle\,|1\,m_2 + 1\rangle$$
$$+ \hbar^2\,m_1\,m_2\,|1\,m_1\rangle\,|1\,m_2\rangle. \tag{28.26}$$

There are really only a few choices we need to consider, since we want $m = m_1 + m_2 = 1$, we can only have $m_1 = 1$, $m_2 = 0$ or $m_1 = 0$, $m_2 = 1$,

$$\mathbf{S}^1 \cdot \mathbf{S}^2\, |1\,1\rangle\,|1\,0\rangle = \hbar^2\,|1\,0\rangle\,|1\,1\rangle$$
$$\mathbf{S}^1 \cdot \mathbf{S}^2\, |1\,0\rangle\,|1\,1\rangle = \hbar^2\,|1\,1\rangle\,|1\,0\rangle. \tag{28.27}$$

Now, it is clear that we must have a special combination, since the action of the dot product on $|1\,1\rangle\,|1\,0\rangle$ yields $|1\,0\rangle\,|1\,1\rangle$ and vice versa. Take the full S^2 operator:

$$S^2\, |1\,1\rangle\,|1\,0\rangle = 2\,\hbar^2\,1\,(1 + 1)\,|1\,1\rangle\,|1\,0\rangle + 2\,\hbar^2\,|1\,0\rangle\,|1\,1\rangle$$
$$S^2\, |1\,0\rangle\,|1\,1\rangle = 2\,\hbar^2\,1\,(1 + 1)\,|1\,0\rangle\,|1\,1\rangle + 2\,\hbar^2\,|1\,1\rangle\,|1\,0\rangle. \tag{28.28}$$

In order to form an eigenstate, we take $\chi \equiv \alpha\,|1\,1\rangle\,|1\,0\rangle + \beta\,|1\,0\rangle\,|1\,1\rangle$, then

$$S^2\,\chi = 2\,\hbar^2\,(2\,\alpha\,|1\,1\rangle\,|1\,0\rangle + \alpha\,|1\,0\rangle\,|1\,1\rangle + 2\,\beta\,|1\,0\rangle\,|1\,1\rangle + \beta\,|1\,1\rangle\,|1\,0\rangle)$$
$$= 2\,\hbar^2\,((2\,\alpha + \beta)\,|1\,1\rangle\,|1\,0\rangle + (2\,\beta + \alpha)\,|1\,0\rangle\,|1\,1\rangle) \tag{28.29}$$

and this will be proportional to χ itself provided $2\alpha + \beta = \alpha$ or $\beta = -\alpha$, so we take, with normalization,

$$\chi = \frac{1}{\sqrt{2}}\left(|1\,1\rangle\,|1\,0\rangle - |1\,0\rangle\,|1\,1\rangle\right) \tag{28.30}$$

which has $S^2\,\chi = 2\,\hbar^2\,\chi$, so is (correctly) spin $s = 1$, and has z-component $S_z\chi = 1\,\hbar\,\chi$.

Finally, we can act on this state with $S_- = S_-^1 + S_-^2$ to get the desired $|1\,0\rangle$ state:

$$S_-\chi = \frac{1}{\sqrt{2}}\left(\hbar\sqrt{2}\,|1\,0\rangle\,|1\,0\rangle - \hbar\sqrt{2}\,|1\,-1\rangle\,|1\,1\rangle + |1\,1\rangle\,\hbar\sqrt{2}\,|1\,-1\rangle - |1\,0\rangle\,\hbar\sqrt{2}\,|1\,0\rangle\right) \tag{28.31}$$

suggesting that the final state of interest, normalized, is

$$|1\,0\rangle = \frac{1}{\sqrt{2}}\left(-|1\,-1\rangle\,|1\,1\rangle + |1\,1\rangle\,|1\,-1\rangle\right). \tag{28.32}$$

28.4 There's Got to be A Better Way!

There is: Given a two spin state combination $|s_1\,m_1\rangle\,|s_2\,m_2\rangle$, the composite state that is an eigenstate of S^2 and S_z, $|s\,m\rangle$ is given by the linear combination:

$$|s\,m\rangle = \sum_{m_1+m_2=m} C^{s_1\,s_2\,s}_{m_2\,m_2\,m}\,|s_1\,m_1\rangle\,|s_2\,m_2\rangle, \tag{28.33}$$

precisely the sort of linear combination we had above – we sum over states that give the target m (i.e. take all m_1 and m_2 such that $m_1 + m_2 = m$) and weight them so as to pick out the appropriate terms for pure S^2 eigenstate with target s. The coefficients, which we calculated by hand above, are called "Clebsch-Gordon" coefficients. For our case with $s = 1$, $m = 1$, which we calculated first, the relevant coefficients are: $\frac{1}{\sqrt{2}}$ for $|1\,1\rangle\,|1\,0\rangle$ and $-\frac{1}{\sqrt{2}}$ for $|1\,0\rangle\,|1\,1\rangle$, leading to

$$|1\,1\rangle = \frac{1}{\sqrt{2}}\left(|1\,1\rangle\,|1\,0\rangle - |1\,0\rangle\,|1\,1\rangle\right). \tag{28.34}$$

The coefficients for the $|1\,0\rangle$ state are $\frac{1}{\sqrt{2}}$ for $|1\,1\rangle\,|1\,-1\rangle$ and $-\frac{1}{\sqrt{2}}$ for $|1\,-1\rangle\,|1\,1\rangle$, giving

$$|1\,0\rangle = \frac{1}{\sqrt{2}}\left(|1\,1\rangle\,|1\,-1\rangle - |1\,-1\rangle\,|1\,1\rangle\right). \tag{28.35}$$

Homework

Reading: Griffiths, pp. 178–184. Note that the homework is due on Monday, April 12th, 2010.

Problem 28.1

Griffiths 4.31. Calculating the spin 1 operators in matrix form.

Problem 28.2

Griffiths 4.33. A time-varying magnetic field. Return to the full time-dependent form of Schrödinger's equation: $H\,\xi(t) \;=\; i\,\hbar\frac{d}{dt}\,\xi(t)$, and be careful with your initial state.