We'll finish with some final comments on the statistical interpretation of kets like $|\Psi\rangle$, both in discrete (but infinite) vector spaces and continuous infinite spaces. In particular, we'll discuss the role of commutators and observables, and look at the implications of vanishing commutator for observables.
Homework

The reading is meant to solidify some of the Bra-ket ideas from class – we’ll return to uncertainty principles on Friday.

Reading: Griffiths, pp. 118–123.

Problem 19.1

Consider the two matrices:

\[ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \] (19.1)

These clearly commute: \([A, B] = 0\), since \(A\) is the identity matrix. This problem is meant to show you that we cannot take just any eigenvectors of \(A\) and expect them to be eigenvectors of \(B\).

a. Show that \(v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) and \(v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\) are eigenvectors of \(A\), and find the eigenvalues. Are these eigenvectors of \(B\)?

b. Find the eigenvectors and eigenvalues of \(B\), show that these are eigenvectors of \(A\).

Conclusion: We can find simultaneous eigenvectors of \(A\) and \(B\) (since they commute), but we are not guaranteed that any set of eigenvectors are simultaneous eigenvectors.

Problem 19.2

Show that an operator \(\hat{Q}\) that is anti-Hermitian:

\[ \int_{-\infty}^{\infty} \psi^*(x) \hat{Q} \psi(x) \, dx = -\int_{-\infty}^{\infty} \left( \hat{Q} \psi(x) \right)^* \psi(x) \, dx \] (19.2)

has purely imaginary expectation values: \(\langle \hat{Q} \rangle\) is imaginary.

Problem 19.3

Our choice of basis is important when computing expectation values (although, of course, the expectation value itself is independent of the basis
in which we compute it), and, even earlier, when we solve Schrödinger’s
equation (think of the position versus momentum representation). Suppose
we take the abstract form of Schrödinger’s equation:
\[ \hat{H} |\Psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle \]  (19.3)
and expand |Ψ(t)\rangle in the natural basis defined by \(\hat{H}\) (i.e. it’s eigenstates).

a. Assume (because \(\hat{H}\) is Hermitian) that we have a complete, con-
tinuous set of eigenstates of \(\hat{H}\) indexed by \(k\):
\[ \hat{H} |\psi_k\rangle = E(k) |\psi_k\rangle \]  (19.4)
where \(E(k)\) is a real number that is a function of \(k\). In addition, we know
that \(\langle \psi_{k'} | \psi_k \rangle = \delta(k-k')\). The expansion, as a function of time of a general
state |Ψ(t)\rangle can be accomplished using the time-dependent decomposition
coefficients \(\phi(k,t)\) in:
\[ |\Psi(t)\rangle = \int_{-\infty}^{\infty} \phi(k,t) |\psi_k\rangle \, dk \]  (19.5)
(so that the decomposition varies in time, but at each time \(t\), we can
represent the ket |Ψ(t)\rangle in terms of the basis |ψ_k\rangle). Input this into (19.3),
and simplify.

b. Now hit both sides of your result from part a. with \(\langle \psi_{k'} |\) and use the
orthonormality relation to solve for \(\phi(k',t)\).