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The main goal of this paper is to sharpen our understanding of what is at stake between two opposing philosophical views, or orientations, on certain issues within and related to mereology. On the one hand, there is a view that reality includes a great deal of *natural mereological structure*, which must be discovered (at least partly) by empirical means, and for which there is no *a priori* reason to think that it will fit any neat formal pattern. Crudely, we may take this first view to be the view that x is part of y if and only if y is an organic or natural union which x partakes in. *Perhaps* the parthood relation has some neat formal properties like transitivity and anti-symmetry, perhaps not; investigation is required. Moreover, it is far from evident that every arbitrary collection of objects constitutes a natural unity, so there probably are many collections for which there is nothing that deserves to be called the “mereological sum” of this collection of objects. Broadly, we should leave it to empirical (natural) science to settle which natural units there are, and what the overall structure of the parthood relation “looks like.”

On the other hand, there is a view that there is an *a priori*¹ science of mereology whose truths reveal a great deal about the overall *pattern* of part-whole connections in the universe. Crudely, we may take this view to be that Classical Mereology (or some similar formal theory) gives the one true theory of the part-whole relation. Very broadly, while the first view might be associated with Aristotle, the second might be associated with more modern figures like Quine and Lewis (though anticipations of it can be found in Descartes and Hume, and elsewhere in the early modern period). As Lewis writes: “I myself take [Classical Mereology] to be perfectly understood, unproblematic, and certain.”² Let us call this second type of view “formalistic.”

Modern proponents of the first type of view—let us call it “naturalistic”—include van Inwagen (according to [5], there are partless simples and mereological fusions of partless simples that are jointly caught up in a life;

¹If we reject a sharp distinction between *a priori* and not, in favor a graduated distinction, then we may substitute “very close to as *a priori* as it gets” for “*a priori*” here.

²[8] p. 75.

there is nothing else) and Koslicki (according to [6], whether some material things have a fusion turns on whether they realize a structure); Kit Fine might also be suggested, but is harder to place (see [1], [2], and elsewhere).

Imagine now a third party to a dispute between a proponent of a naturalistic view and a proponent of a formalistic view, who wishes to make a kind of peace between them by arguing that their differences are not ultimately as great as might at first appear. The general strategy the third party employs is to try to show that for each of the two disputants, the third party can find, within the things and structure the disputant believes in, a kind of “image” of the things and structure the other disputant believes in. It may be that, after “looking at the world from each other’s point of view” the disputants find that the differences between them are negligible; or, perhaps more likely, that the exact nature of the disagreement between them is made sharper by getting clearer on why the differences, despite the existence of the “images,” are not negligible.

A simplification: sets as natural kinds

A comparison to a somewhat simpler dispute will help make clear what I have in mind. Consider a dispute between two philosophers, the first of whom, in “naturalistic” fashion, holds that some but not all classes of material objects correspond to *natural kinds* (e.g., the class of all dogs corresponds to a natural kind, but the class of all dogs that are in a country whose name begins with “E” does not). The naturalistic philosopher believes in arbitrary classes of things, and *in addition*, a few *kinds* of things. The second, “formalistic,” philosopher is skeptical of the existence of *kinds* above and beyond the classes themselves. Now imagine a third party who gets *both* philosophers to agree that every class (of material objects) corresponds to one and only one *set* of objects (perhaps they take a class to itself be a set, or perhaps they take a class to be a mere plurality and a set to be a single thing). The third party then proposes that the naturalistic philosopher might see the naturalistic one as simply concerned to deny that there are any further entities that “collect” material objects, above and beyond the sets, so that if there are natural kinds, they are just sets. Meanwhile, the formalistic philosopher might see the naturalistic one as holding that

among all the many sets of material objects, some are special, and deserve to be singled out as “natural.”

If the formalistic philosopher agrees that some sets are especially natural, and the naturalistic one does not think that an *ontology* of kinds is necessary, in addition to the distinction between natural and unnatural sets, then it is unclear that the two really disagree on anything that matters. The situation can be compared to the dispute about universals, between David Armstrong and David Lewis, as portrayed by Lewis in [7]. Lewis (in the role corresponding to our formalistic philosopher) at first wants to deny that there are universals, in addition to arbitrary collections of possibilities. But he then comes to recognize that the whole system, advocated by Armstrong, of a sparse ontology of universals,³ together with certain features of them (their direct relations to laws of nature, to objective similarity, etc.) has great theoretical utility. But instead of adding a superstratum of universals to his arbitrary collections, Lewis proposes that all of the theoretical work that universals need to do can be done by the collections *together with* a crucial distinction between *perfectly natural* collections and other collections. One might say that his ontology is formalistic, but his ideology is naturalistic.

The disagreement between our two philosophers thus might be merely superficial: they might ultimately agree in ontology (sets alone, no other ontological type required) and theoretical ideology (there is an extremely important natural/non-natural distinction among sets). The disagreement might instead be deep, perhaps because the naturalistic philosopher takes himself to have good reasons to believe that kinds are not certain special sets, or perhaps because the formalistic philosopher takes the distinction between natural and unnatural to be unacceptable, either in general, or in its application to sets. Or again, perhaps both philosophers agree that sets do not change their members, and the first philosopher holds that natural kinds do change their members: e.g., the kind *dog* loses a member each time a dog dies. Then there appears to be a good reason to think that the set is intrinsically, hopelessly, unsuited to play the theoretical role required of the kind.

For our purposes, it is worth dwelling on this story just a little longer. While it is not implausible that sets do not change their members, while

³“Sparse,” because not every arbitrary collection corresponds to a universal.

kinds do, it is also not implausible to think that this is a mere appearance of difference, resulting from typical ways of talking, rather than the natures of the things themselves. For it may be agreeable to both philosophers that a set has its members “eternally,” so that the set of all dogs that ever exist currently has members that do not presently exist. Set membership, on this view, doesn’t occur, or relate a member to a set, at one time rather than another; instead, it happens timelessly. Yet if this is the case there is still a reasonable notion of a set s losing a member x at a time t : x might be a member of s such that [x exists over a long span of time up to t , and x does not exist after t]. Once it is recognized that both (1) set membership is an eternal affair (so that $x \in s$ either once-and-for-all or never); and (2) nonetheless, there is a reasonable derivative notion of membership-at-a-time ($x \in_t s$ iff [$x \in s$ and x exists at t]) it is less clear that the fact that we tend to think of the natural kind *dog* as subject to membership-change while we tend to think of sets as membership-stable is a good reason to think that sets and natural kinds are different types of things. For it may be that when we think of the changeable membership relation on natural kinds we are really just thinking of the derivative changeable membership relation on (natural) sets.

The philosophers debating on natural kinds might continue to disagree. The naturalistic one might say that sets are ineligible to be kinds for another reason: they have the wrong *spatial* properties. The natural kind *dog* might be something that exists on earth, while the set of dogs exists nowhere or everywhere. But again, there is reason to wonder if this is a genuine difference rather than an appearance. For we may certainly define a notion of location for a set at a time that will behave, one might think, much like the notion of location for a kind does: the set will be located, at a time, wherever its members that exist at that time are. More precisely, we will need to say something like: the location of s at t is the union of the regions occupied at t by the members of s that exist at t . It is worth noting that part of what makes this particular definition work is that it is relatively uncontroversial that *regions* amalgamate in a natural way: for any collection of regions, there is the union of those regions, basically a region that you partly occupy if and only if you partly occupy any of the regions in the collection.

Now, it is possible to insist that such a notion of the “location” of a set is somehow second-rate (“unnatural” or “fake” or “merely derivative,”

etc.), while the notion of the “location” of a kind is first-rate, not second-rate. But it is unclear how such an asymmetric ranking of the two notions of location can be justified.

Similarly, if the naturalistic philosopher protests that kinds are made of matter, while sets are not, we might wonder why a well-defined notion of the material content of a set (if we can find one) is second-rate. Say that a set is “perfectly materialistic” if it is non-empty and every one of its members is made of matter. If we may suppose that for any bits of matter, there is some matter that functions as the “union” of those bits, in much the way that for any collection of regions of space there is a union of the regions, then we may say that a perfectly materialistic set is “made of” exactly the union of the bits of matter that make up its members.

It is not obvious how far such strategies can actually work to remove apparent differences between the set of dogs and the natural kind *dog*. But the basic point should now be clear enough: that the dispute between the two philosophers who seem to disagree about natural kinds might well turn out to be a shallow or merely verbal dispute, since it may turn out that each philosopher believes in a system of items and features of those items, a system that plays the theoretical role that the whole system of natural kinds is supposed to play.

Sets as things

Now to return to the main theme: the suggestion of this paper is that the dispute between the “natural unities” mereologist and the “mathematical pattern” mereologist may turn out to be similarly largely shallow or verbal. In particular, the suggestion will be that if the naturalistic mereologist agrees to the existence of arbitrary *sets* of the material objects he or she already embraces, while the formalistic mereologist agrees to a crucial distinction between natural and non-natural objects and sets of objects, the two may equally regard the whole of reality to consist of a formally well-behaved pattern of objects and sets of objects (a pattern whose global properties are what the formalistic mereologist was always emphasizing), together with an important, formally unpredictable, natural/non-natural distinction among the nodes in this pattern (which the naturalistic mereologist was always emphasizing).

To illustrate a little: where the formalistic mereologist takes there to be a fusion of all objects which are either cats or dogs, the naturalistic one takes there to be the set of all things that are either cats or dogs. Now, both agree that the set exists, and we take it that it is negotiable that the set might inherit a *location*, and other minimal physical properties, from its members. But then how different is the set, as seen from the point of view of the naturalist, from the fusion, as seen from the point of view of the formalist? By downplaying the differences, we hope to make good on our suggestion that the mutually acceptable set can play the role of the fusion. Assuming that this works for this particular object (the fusion), our main task is to show how to coordinate things so as to make an entire formalistic network of objects, and part-whole relations among them, mutually agreeable. The mutually accepted network will have exactly the formal character that the formalistic mereologist emphasized; yet the naturalistic philosopher will still maintain that there is a special *natural* sub-network of the larger, formally well-behaved one, with natural objects as nodes, linked by a natural sub-relation of the larger part-whole relation.

The rest of this paper is concerned with some technical details involved with fleshing out this suggestion, particularly from the point of view of the naturalistic mereology. The main project at hand is of this form: assuming nothing formally about the most basic, *given* system of objects D and primitive “natural” part-whole relation \mathbf{N}_0 on them, what needs to be done, using nothing more than set theory together with the given objects and relation, to construct on and around it a formally “well-behaved” system of objects H and defined part-whole relation \leq on them? We wish to “preserve” as much structure as possible, with D a subset of H and \mathbf{N}_0 a sub-relation of \leq , and such that the relation \leq , when restricted to its sub-domain D , should be identical with, or at least very closely related to, \mathbf{N}_0 . To make this project more exact, we will take the notion of being “well-behaved” to be the notion of “obeying the laws of Classical Mereology,” so that what we are up to is finding a transformation Ψ , that could in principle be applied to any relational structure $\langle X, \mathbf{R} \rangle$, so that

$$\Psi(\langle X, \mathbf{R} \rangle) = \langle X', \mathbf{R}' \rangle$$

has exactly the formal structure that Classical Mereology requires; that is, $\langle X', \mathbf{R}' \rangle$ is guaranteed to be a model of Classical Mereology, no matter what X and \mathbf{R} are.

What makes the project formally non-trivial is that there are basically two sorts of formal task here, that tend to work against one another, but must be executed simultaneously. The first task is this: given a “natural” part-whole relation \mathbf{N}_0 and its “natural” domain D , *extend* the relation — that is, add relational “links” to \mathbf{N}_0 , among things already present in D — in such a way that the resulting relation is formally well-behaved in the sense of possessing such features as reflexivity, transitivity, and obeying the *strong supplementation*⁴ constraint of Classical Mereology. The second task is to add objects to the “natural” domain D (together with relational links) so as to provide mereological fusions for arbitrary non-empty subsets of this domain.

We would be on our way to executing the second task, if we were to imitate in a straightforward way what we considered saying about natural kinds above: “let us add to D every non-empty subset of D , and count members as parts.” Thus we would get a candidate for the mereological fusion of all dogs: the set of all dogs would now be counted as a material object, alongside the dogs, and each dog would count as a part of it. Many objections to so counting the set can be met with, as discussed above. But this way of executing the second task has made it harder to execute the first task. For example, our new relation will not be transitive on its domain, since a given dog’s foot will not be counted as a part of the fusion of all dogs. Moreover, we may have “too many things” in some cases, playing the same formal role: for example, if p is the set of parts (in the original, given sense of part) of a dog d , then both d and p are suited to play the formal role of being the mereological fusion of the members of p .

Thus the non-trivial formal difficulty is in executing both tasks simultaneously. But it can be done, in a fairly natural way. While the formal device explored here is, it is hoped, sufficient to give a “proof of concept” for the more general philosophical idea, it is really only a first step, as there are a number of questions one might raise about it that we will not have the space to discuss. A couple will be touched on briefly at the end of the paper, once the device is in view.

⁴See below for a formally exact statement of this constraint.

Overview of the formal device

Here is a brief informal sketch of the technique. We begin with some *natural objects* (to be thought of as concrete *natural units* on the model of the naturalistic mereology) and a given part-whole relation on them; call the set of these objects the *natural domain* and the relation the *natural part-whole relation*. Then we take the reflexive and transitive closure of the natural part-whole relation; next we *extend the domain* by adding non-empty, non-singleton *sets* of the members of the natural domain. We then extend the relation further, reaching a relation on the extended domain that is logically guaranteed to almost satisfy CM. *Almost*, because, in a very clear sense, the *only* possible failing is that the resulting relation might not be anti-symmetric. In the final stage, we *restrict the domain and relation* that resulted from the composite of our previous transformations, basically choosing (in a principled way) one “representative” from each cluster of items that contravene anti-symmetry, thus guaranteeing that we move from *almost* satisfying CM to actually satisfying it.

An interesting feature of the general transformation is this: if we start with a domain and relation that satisfies CM, the construction winds up exactly where it started: the combined effect of our sequence of transformations will be nothing at all. CM is, structurally, a “fixed-point” of the construction.

Formalization

We turn to the technical details of the transformation; the discussion assumes only an elementary acquaintance with logic and set theory, and should be accessible to anyone interested in the formal details of Classical (and other) mereologies.

We will be discussing various transformations on *relational structures*, that is, ordered pairs $\langle X, \mathbf{R} \rangle$, where X is a set and \mathbf{R} is a relation on that set (the *carrier set*). Relations are simply sets of ordered pairs, and what it means that \mathbf{R} is a relation *on* X is just that for every ordered pair $\langle x, y \rangle$ in the relation, $x \in X$ and $y \in X$, or, to put it another way, $\mathbf{R} \subseteq X \times X$. We will often write ‘ $x \mathbf{R} y$ ’ for ‘ $\langle x, y \rangle \in \mathbf{R}$ ’; we will also say ‘ x bears \mathbf{R} to y ’ for this.

Another notion we will want is the notion of the *restriction* of a relation to a given set: if \mathbf{R} is a relation and Y is a set, then $\mathbf{R} \upharpoonright Y$ is the relation $\{\langle x, y \rangle : x \mathbf{R} y \text{ and } x, y \in Y\}$.

We will focus on “part-like” relations and structures, and a particular sequence of transformations on them. But the transformations we consider can be defined in a general way, independent of their application here; we will consider the general definitions as well as the application.

The first transformation, Φ_r , is simply to take the reflexive closure of a relation (on the carrier set):

$$\Phi_r(\langle X, \mathbf{R} \rangle) = \langle X, \mathbf{R} \cup \{\langle x, x \rangle : x \in X\} \rangle.$$

Clearly, if \mathbf{R} is itself reflexive, then $\Phi_r(\langle X, \mathbf{R} \rangle) = \langle X, \mathbf{R} \rangle$. So Φ_r is *self-fixing*: for any relational structure \mathfrak{B} , $\Phi_r(\Phi_r(\mathfrak{B})) = \Phi_r(\mathfrak{B})$.

The next transformation, Φ_t , takes the transitive closure of the given relation. Given $\langle X, \mathbf{R} \rangle$, say that \mathbf{S} *transitively extends \mathbf{R} within X* if $\mathbf{S} \subseteq X \times X$, $\mathbf{R} \subseteq \mathbf{S}$, and \mathbf{S} is transitive. \mathbf{R}^t is then $\bigcap \{\mathbf{S} : \mathbf{S} \text{ transitively extends } \mathbf{R} \text{ within } X\}$, and we define Φ_t so that

$$\Phi_t(\langle X, \mathbf{R} \rangle) = \langle X, \mathbf{R}^t \rangle.$$

The transitive closure of a relation is itself transitive, since the intersection of a set of transitive relations is itself transitive. If \mathbf{R} is itself transitive, then $\Phi_t(\langle X, \mathbf{R} \rangle) = \langle X, \mathbf{R} \rangle$. So Φ_t is also self-fixing. Further, $\Phi_t(\Phi_r(\mathfrak{B})) = \Phi_r(\Phi_t(\mathfrak{B}))$.⁵

To begin our discussion of the application, let D be the set of natural objects. (We assume that they form a set.) We may allow that there are many specific part-whole relations on D ; let us define

$$x \mathbf{N}_0 y$$

so that for $x, y \in D$, $x \mathbf{N}_0 y$ just in case x bears one of these relations to y .

⁵ One can get an especially clear view of the effect of Φ_t by considering how it can be *built up* from iterated application of a simpler transformation. Define Φ_{t_0} so that

$$\Phi_{t_0}(\langle X, \mathbf{R} \rangle) = \langle X, \mathbf{R} \cup \{\langle x, z \rangle (\in X \times X) : \exists y (x \mathbf{R} y \wedge y \mathbf{R} z)\} \rangle.$$

$\Phi_{t_0}(\mathfrak{B})$ is a first approximation of $\Phi_t(\mathfrak{B})$; a second approximation is $\Phi_{t_0}(\Phi_{t_0}(\mathfrak{B}))$. One can show that $\Phi_t(\mathfrak{B})$ is the “limit” of the approximations. More precisely: let \mathfrak{B}^0 be $\mathfrak{B} = \langle X, \mathbf{R} \rangle$ and let \mathfrak{B}^{i+1} be $\Phi_{t_0}(\mathfrak{B}^i)$. Let \mathbf{R}^i be the relation in \mathfrak{B}^i . Then \mathbf{R}^t , the relation of $\Phi_t(\mathfrak{B})$, is the relation

$$\{\langle x, y \rangle (\in X \times X) : \exists i \in \mathbb{N} \langle x, y \rangle \in \mathbf{R}^i\}.$$

\mathbf{N}_0 is the resulting generalized *natural* part-whole relation.

Formally, we make no assumptions whatever about \mathbf{N}_0 : $\langle D, \mathbf{N}_0 \rangle$ is an arbitrary non-empty relational structure (a non-empty set with a relation on it). Informally, we will use natural examples like John's foot being part of John.

Now let $\langle D, \mathbf{N} \rangle$ be $\Phi_r(\Phi_t(\langle D, \mathbf{N}_0 \rangle))$, so that \mathbf{N} is the relation that arises from taking the transitive closure of \mathbf{N}_0 and adding reflexivity.

Our next general transformation Φ_1 is somewhat complicated. Say that a set is *suitable* if it has two or more members. Given $\langle X, \mathbf{R} \rangle$, let A be the set of all suitable subsets of X , and let $B = X \cup A$. Let \mathbf{S} be the relation on B that holds of x and y just in case

$$x \mathbf{R} y, \text{ or } x \in y, \text{ or } x \subseteq y.$$

Then let $\Phi_1(\langle X, \mathbf{R} \rangle) = \langle B, \mathbf{S} \rangle$. Clearly Φ_1 is not self-fixing; in fact, almost the opposite: provided the carrier set X itself is suitable, $\Phi_1(\langle X, \mathbf{R} \rangle) \neq \langle X, \mathbf{R} \rangle$.

Let $\langle E, \mathbf{P}_0 \rangle$ be $\Phi_1(\langle D, \mathbf{N} \rangle)$, i.e., $\Phi_1(\Phi_r(\Phi_t(\langle D, \mathbf{N}_0 \rangle)))$. Then we can show that $x \mathbf{P}_0 y$ if and only if one of the following holds:

$$x \mathbf{N} y, \text{ or } x \in y, \text{ or } x \subseteq y.$$

Each of the three disjuncts excludes the other two.

Let E° be the set of suitable subsets of D , so that $E = D \cup E^\circ$ and $D \cap E^\circ = \emptyset$. Let $\langle E, \mathbf{P} \rangle$ be $\Phi_t(\langle E, \mathbf{P}_0 \rangle)$. Then one can confirm that $x \mathbf{P} y$ just in case either:

$$x \mathbf{P}_0 y \text{ or } x \in D \text{ and } y \in E^\circ, \text{ and there is some } b \in y \text{ such that } x \mathbf{P} b.$$

To show this, consider what was added when we applied Φ_t to $\langle E, \mathbf{P}_0 \rangle$ (show the easy Lemmas 1 and 2 below first). This shows that to define \mathbf{P} , we could have used these clauses instead of Φ_t , in our particular application. Also, instead of applying Φ_r and Φ_t to get \mathbf{N} from \mathbf{N}_0 first, we could have applied Φ_1 directly to $\langle D, \mathbf{N}_0 \rangle$ and then applied Φ_r and Φ_t (or the above clauses); the result would be the same.

Let us observe some more features of \mathbf{P} . First, some informal examples: let *foot* be John's foot and *hand* be John's hand. Then

$$\textit{foot} \mathbf{P} \textit{John}$$

John \mathbf{P} { John, the Eiffel Tower }
 (and hence) *foot* \mathbf{P} { John, the Eiffel Tower }.

But

it is not the case that { *hand, foot* } \mathbf{P} John.

Second, some structural features. \mathbf{P} has a “top” element, namely D : every member of E bears \mathbf{P} to D . So everything in the wider domain is “part of” the set of all objects (the narrow domain). Clearly, \mathbf{P} is reflexive and transitive (on E). A very important feature we will use later is this: if some $b \in D$ bears \mathbf{P} to some $i \in E^\circ$, then b bears \mathbf{P} to some $c \in i$ (in fact, $b \mathbf{N} c$). That is,

Lemma 1 $(b \in D \wedge i \in E^\circ) \rightarrow (b \mathbf{P} i \rightarrow \exists c \in D(c \in i \wedge b \mathbf{P} c))$.

Also note

Lemma 2 $(i \in E^\circ \wedge j \in E^\circ) \rightarrow (i \mathbf{P} j \leftrightarrow i \subseteq j)$
 and $(i \in E^\circ \wedge b \in D) \rightarrow \neg i \mathbf{P} b$.

\mathbf{P} is in the direction of the Classical Mereologist’s part-whole relation: the set of some objects from D is playing something like the role of the mereological fusion of its members, since every part (in the sense of \mathbf{N}_0) of every member bears \mathbf{P} to the set. But this approximation, to the “fusion” of a set of things that happen to be parts of something x , may not bear \mathbf{P} to x , so we are not there yet. For example, if x is the set of John’s parts, $x \in E^\circ$ (assuming John has more than one part) and it is not the case that $x \mathbf{P}$ John.

Minimal upper bounds and complements

The next transformation takes us much closer. Given any structure $\langle X, \mathbf{R} \rangle$, define the relation $\circ_{\mathbf{R}}$ (\mathbf{R} -overlap) on X as:

$$(\forall x, y \in X) (x \circ_{\mathbf{R}} y \leftrightarrow \exists z(z \mathbf{R} x \wedge z \mathbf{R} y)).$$

Then define \mathbf{S} as: $x \mathbf{S} y$ iff $\forall z (z \circ_{\mathbf{R}} x \rightarrow z \circ_{\mathbf{R}} y)$. Finally, define Φ_o so that

$$\Phi_o(\langle X, \mathbf{R} \rangle) = \langle X, \mathbf{S} \rangle.$$

Let $\langle E, \sqsubseteq \rangle$ be $\Phi_o(\langle E, \mathbf{P} \rangle)$ i.e., $\Phi_o(\Phi_t(\Phi_1(\Phi_r(\Phi_t(\langle D, \mathbf{N}_0 \rangle))))$. Let us notate the relation of \mathbf{P} -overlap as ∇ . Consider again { *hand, foot* }; temporarily

call it i . Given $x \in E$, if $x \nabla i$, then there is a $w \in E$ that bears \mathbf{P} to x and to i . We argue now that there is a $b \in D$ such that b bears \mathbf{P} to w and either to *hand* or to *foot*. If $w \in D$ then let $b = w$ (see Lemma 1). If $w \in E^\circ$, $w \subseteq i$, so $w = i$ (since i is a doubleton), and let $b = \textit{hand}$. But b then bears \mathbf{P} to John; and b bears \mathbf{P} to x (since $b \mathbf{P} w$ and $w \mathbf{P} x$); thus, $x \mathbf{P}$ -overlaps John. This all shows that

$$\{ \textit{hand}, \textit{foot} \} \sqsubseteq \text{John}.$$

Let us now consider the structural features of \sqsubseteq . It is easy to see from its definition (without even knowing what ∇ means) that \sqsubseteq is reflexive and transitive. We also have

Lemma 3 *If $\Phi_o(\langle X, \mathbf{R} \rangle) = \langle X, \mathbf{S} \rangle$, then, provided that \mathbf{R} is transitive, $(\forall x, y \in X) (x \mathbf{R} y \rightarrow x \mathbf{S} y)$.*

In particular, $(\forall x, y \in E) (x \mathbf{P} y \rightarrow x \sqsubseteq y)$.

We now are much closer to the behavior of fusions, since we have

$$\{ x : x \mathbf{P} \text{John} \} \sqsubseteq \text{John}.$$

To show how close we are will require some work. First, we will define a sum-like notion. Given a non-empty $X \subseteq E$, let

$$\sigma'(X) = \{ b \in D : (\exists y \in X) b \mathbf{P} y \}.$$

$\sigma'(X)$ is obviously non-empty. It is a singleton if and only if X is a singleton of a \mathbf{P} -atom (a member of D that nothing else bears \mathbf{P} to); and then $\sigma'(X) = X$. In this case, $\sigma'(X) \notin E$; otherwise $\sigma'(X) \in E$. Accordingly, let

$$\sigma(X) = \sigma'(X) \text{ if } \sigma'(X) \in E; \text{ otherwise, let } \sigma(X) \text{ be the one member of } \sigma'(X).$$

We will prove that $\sigma(X)$ is a minimal upper bound on X : every member of X bears \sqsubseteq to it, and it bears \sqsubseteq to any such thing.

Lemma 4 $(\forall x, y \in E) (x \nabla y \rightarrow (\exists b \in D) (b \mathbf{P} x \wedge b \mathbf{P} y))$

Lemma 5 $(\forall b \in D)(\forall i \in E^\circ) (b \nabla i \rightarrow (\exists c \in i) b \nabla c)$

Both of these Lemmas are easy to confirm from Lemmas 1 and 2.

Lemma 6 $(\forall X \subseteq E) : X \neq \emptyset \rightarrow (\forall x \in X) x \sqsubseteq \sigma(X)$

Proof: Let $x \in X$. Then, if $y \nabla x$, by Lemma 4, we have a $b \in D$ with $b \mathbf{P} x$ and $b \mathbf{P} y$. By the definition of $\sigma(X)$, $b \subseteq \sigma(X)$ (i.e., either $b = \sigma(X)$ or $b \in \sigma(X)$); so $b \mathbf{P} \sigma(X)$. So $y \nabla \sigma(X)$.

Lemma 7 $(\forall y \in E) ((\forall x \in X) x \sqsubseteq y \rightarrow \sigma(X) \sqsubseteq y)$

Proof: Suppose $(\forall x \in X) x \sqsubseteq y$. Suppose $w \nabla \sigma(X)$. Then, by Lemma 4, we have a $b \in D$ such that $b \mathbf{P} w$ and $b \mathbf{P} \sigma(X)$. By Lemma 1, there must be a $c \in D$ with $c \subseteq \sigma(X)$ such that $b \mathbf{P} c$. Since $c \subseteq \sigma(X)$, for some $x' \in X$ $c \mathbf{P} x'$; by Lemma 3 and our original supposition, $c \sqsubseteq y$. $w \nabla c$, hence $w \nabla y$, and we are done.

Lemmas 6 and 7 together say that $\sigma(X)$ is a minimal upper bound for X , with respect to the \sqsubseteq relation. Formally, define: y is a \sqsubseteq -minimal upper bound on X if and only if

$$(\forall x \in X) x \sqsubseteq y \wedge \forall z ((\forall x \in X) x \sqsubseteq z \rightarrow y \sqsubseteq z).$$

We have now shown

Theorem 1 *For every non-empty $X \subseteq E$, X has a \sqsubseteq -minimal upper bound.*

$\sigma(X)$ plays this role, so $\sigma(X)$ is an approximation of the fusion of X .

Complements

The \sqsubseteq relation on E has even more in common with the classical mereologist's part-whole relation, since it includes what we may call *complements*. Roughly, for almost any object in E , there is a another object that represents "everything else" in E : the complement is "disjoint" from the original, but everything "overlaps" one or the other. The only objects without complements are objects of which everything is already a "part."

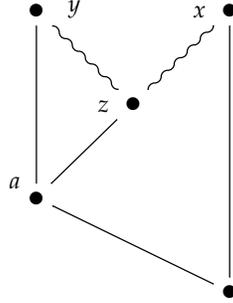
Define the \sqsubseteq -overlap relation (symbolized with \square) as

$$x \square y \leftrightarrow \exists z (z \sqsubseteq x \wedge z \sqsubseteq y)$$

Lemma 8 $(\forall x, y \in E) (x \square y \leftrightarrow x \nabla y)$

Proof: The right-to-left direction is straightforward from Lemma 3. For the left-to-right direction, we give a visual proof. Straight lines represent

holdings of the \mathbf{P} relation from lower to higher, and squiggly lines represent holdings of the \sqsubseteq relation from lower to higher.



a has to exist, since $z \nabla z$ and $z \sqsubseteq y$; but then $a \nabla x$ as well.

In view of Lemma 8, we can interchange \square and ∇ as we please.

Lemma 9 $(\forall x, y \in E)(\forall z \in E)(z \sqsubseteq x \rightarrow z \square y) \rightarrow x \sqsubseteq y$

Proof: Suppose the antecedent and that $w \nabla x$, and let $z \mathbf{P} w$ and $z \mathbf{P} x$. By Lemma 3 and the antecedent, $z \square y$. By Lemma 8, $z \nabla y$, so $w \nabla y$.

We will also want the notions of \mathbf{P} -disjointness and \sqsubseteq -disjointness, where each is non-overlap of the relevant sort. Given Lemma 8 these relations are interchangeable. For notation, set

$$\begin{aligned} x \wr y &\leftrightarrow \neg x \nabla y \text{ (or equivalently)} \\ x \wr y &\leftrightarrow \neg x \square y \end{aligned}$$

Now we find, for almost any member of E , an object that will play the role of its complement. Given $x \in E$, if there is a $y \in E$ with $y \not\sqsubseteq x$, then define

$$x^* = \sigma\{y \in E : y \wr x\}$$

We can use Lemma 9 to show that $\{y \in E : y \wr x\}$ is non-empty: so x^* exists.

Lemma 10 $x \wr x^*$

Proof: Suppose for *reductio* $x \nabla x^*$. Then either $x^* \in D$ (in which case $\{y \in E : y \wr x\}$ was $\{x^*\}$ and it is clear from the definition that $x^* \wr x$) or get a $b \in D$ with $b \mathbf{P} x$ and $b \mathbf{P} x^*$; since $b \mathbf{P} x^*$, get (by Lemma 1) a $c \in x^*$ with $b \mathbf{P} c$. Using the def. of x^* , confirm that $c \wr x$. But $b \mathbf{P} c$ and $b \mathbf{P} x$, so $c \nabla x$; contradiction.

Lemma 11 $y \wr x \rightarrow y \sqsubseteq x^*$

Proof: Suppose $y \wr x$ and $w \nabla y$. Get (by Lemma 4) $b \in D$ with $b \mathbf{P} w$ and $b \mathbf{P} y$. Now if $b \nabla x$ then $y \nabla x$; we supposed not, so $b \wr x$. So $b \sqsubseteq x^*$. So $b \mathbf{P} x^*$, so $w \nabla x^*$.

Lemma 12 $y \wr x^* \rightarrow y \sqsubseteq x$

Proof: Suppose $y \wr x^*$ and $w \nabla y$. Get $b \in D$ with $b \mathbf{P} w$ and $b \mathbf{P} y$. Get that $b \wr x^*$, so $b \not\sqsubseteq x^*$, so it is not the case that $b \wr x$, so $b \nabla x$, and hence $w \nabla x$.

Putting the last three lemmas together, we have that everything that is not all-inclusive has a “ \sqsubseteq -complement” where we define: y is a \sqsubseteq -complement of x if and only if

$$y \wr x \text{ and } \forall z ((z \wr x \rightarrow z \sqsubseteq y) \text{ and } (z \wr y \rightarrow z \sqsubseteq x))$$

Theorem 2 For all $x \in E$, if $\exists y (y \not\sqsubseteq x)$ then x has a \sqsubseteq -complement.

For all, except the all-inclusive $x \in E$, x has at least one complement, and x^* is one.

Anti-symmetry

The relation \sqsubseteq on E is formally very much like the Classical Mereologist’s part-whole relation. For we have shown that \sqsubseteq and E are a relation \mathbf{R} on a set X such that

- (2) \mathbf{R} is transitive.
- (3) All non-empty subsets of X have an \mathbf{R} -minimal upper bound.
- (4) For any member of X , if not everything bears \mathbf{R} to it, then it has a complement.

If a relation \mathbf{R} on a domain X satisfies (2)-(4), then the structure $\langle X, \mathbf{R} \rangle$ satisfies the axioms of Classical Mereology, provided it has two further features: (1) \mathbf{R} is anti-symmetric; and (5) either there is only one member of X or there is no member of X that bears \mathbf{R} to every member of X .⁶

⁶See section 4 of [4]; the five conditions here correspond to the five axioms in the last of the five axiom-sets given there.

The members of E fall into “clusters” of things that bear \sqsubseteq to one another. These are like the equivalence classes of an equivalence relation, except that members of different clusters may (anti-symmetrically) bear \sqsubseteq to one another. If a member of a cluster k bears \sqsubseteq to a member of some other cluster l , then every member of k bears \sqsubseteq to every member of l , and no member of l bears \sqsubseteq to any member of k . There is a simple way to transform the structure $\langle E, \sqsubseteq \rangle$ into a Classical Mereology. We simply treat each “cluster” of things that bear \sqsubseteq to each other as a single element, and let the clusters inherit the other aspects of the \sqsubseteq relation. Formally, for each $x \in E$, define

$$[x] = \{y \in E : x \sqsubseteq y \wedge y \sqsubseteq x\}$$

Let F be $\{y : \exists x \in E y = [x]\}$. For $[x], [y] \in F$, with $x, y \in E$, define

$$[x] \leq^F [y] \text{ if and only if } \exists z \in [x] \exists w \in [y] z \sqsubseteq w.$$

We can think of this as an instance of a general transformation Φ_a taking us from $\langle E, \sqsubseteq \rangle$ to $\langle F, \leq^F \rangle$; the definition is confined to a footnote.⁷

Lemma 13 \leq^F on F is reflexive, anti-symmetric, and transitive.

Suppose $X \subseteq F$ is non-empty. Let z be $\{c \in E : [c] \in X\}$. Theorem 1 tells us that z has at least one \sqsubseteq -minimal upper bound d . Let $\bigvee X$ be $[d]$.

Lemma 14 $\bigvee X$ is a least upper bound for X (in F).

That is, for every $x \in X$, $x \leq^F \bigvee X$, and, for any $y \in F$, if every $x \in X \leq^F y$, then $\bigvee X \leq^F y$. This is straightforward to show. (We call this a “least” upper bound since, because of anti-symmetry, it is unique.)

Lemma 15 If F has more than one element, then there is no $x \in F$ such that $\forall y \in F, x \leq^F y$.

⁷Given any structure $\langle X, \mathbf{R} \rangle$, let $A = \mathcal{P}(X)$. Given any $x \in X$, let $[x] = \{y \in X : x \mathbf{R} y \wedge y \mathbf{R} x\}$. Let B be $\{e \in A : \exists x \in X \wedge e = [x]\}$. Let \mathbf{S} be the relation on B defined as follows: for any e and f in B ,

$$e \mathbf{S} f \text{ if and only if } (\exists z \in e)(\exists w \in f) z \mathbf{R} w.$$

Then define $\Phi_a(\langle X, \mathbf{R} \rangle) = \langle B, \mathbf{S} \rangle$. In general, this transformation is much more natural when combined with prior application of Φ_r and Φ_t ; the composite $\Phi_a \circ \Phi_r \circ \Phi_t$ transforms any relational structure into a partial ordering.

Proof: It is clear that $\forall x \in E, x \leq \sigma(E)$, and so $[x] \leq^F [\sigma(E)]$. Now consider any $[x] \in F$ such that $[\sigma(E)] \not\leq^F [x]$. Apply Theorem 2 and get x^* with $x^* \wr x$; hence $x \not\sqsubseteq x^*$. Thus $[x] \not\leq^F [x^*]$.

Finally, suppose that for a given $x \in F$, there is a $y \in F$ with $y \not\leq^F x$. Then there is a \leq^F -complement for x (uniquely so, because of transitivity). Define $x, y \in F$ are \leq^F -disjoint (symbolized \wr^F) as

$$x \wr^F y \text{ if and only if } \neg \exists z \in F (z \leq^F x \wedge z \leq^F y)$$

For $x, y \in F$, define x is a complement of y as

$$x \wr^F y \text{ and } \forall z \in F ((z \wr^F x \rightarrow z \leq^F y) \text{ and } (z \wr^F y \rightarrow z \leq^F x))$$

Lemma 16 *For every $x \in F$, if $\exists y (y \wr^F x)$, then x has a \leq^F -complement.*

Proof: Suppose we have an x as in the antecedent. Then pick some $a \in x$ and consider $[a^*]$.

By the last four Lemmas, we have

Theorem 3 $\langle F, \leq^F \rangle$ is a Classical Mereology.

Now, we can “project” the structure of \leq^F into E by mapping each $f \in F$ to some representative member of it. The set of representatives would be a subset of E , and the restriction of \sqsubseteq to this subset would be isomorphic to \leq^F .

There are at least two fairly natural ways to choose representatives. The first is this: for each $[x] \in F$, we pick $\sigma([x])$. To see that this works, we need to show

Lemma 17 $(\forall x \in E) \sigma([x]) \in [x]$

Proof: Suppose $y \in [x]$. Then, by Lemma 6, $y \sqsubseteq \sigma([x])$. And for all $z \in [x]$, $z \sqsubseteq y$. Hence, by Lemma 7, $\sigma([x]) \sqsubseteq y$.

So now let G be $\{x : x = \sigma(f) \text{ for some } f \in F\}$. Then $G \subseteq E$, and we let \leq^G be $\sqsubseteq \upharpoonright G$. Then $\langle G, \leq^G \rangle$ is isomorphic to $\langle F, \leq^F \rangle$: σ is a one-one map from F onto G , and $f \leq^F g$ iff $\sigma(f) \leq^G \sigma(g)$.

The second, preferred, way to choose representatives that we will consider is to choose the “smallest” representative, if there is one; otherwise choose the “largest,” namely $\sigma([x])$. For each $x \in E$: if $x \cap D = \{b\}$ for some b , then let $\rho([x]) = b$; otherwise, let $\rho([x]) = \sigma([x])$. Let H be $\{x : x = \rho(f) \text{ for some } f \in F\}$. Then $H \subseteq E$, and we let \leq be $\sqsubseteq \upharpoonright H$. Clearly, $\langle H, \leq \rangle$ is also isomorphic to $\langle F, \leq^F \rangle$. So we have:

Theorem 4 $\langle G, \leq^G \rangle$ and $\langle H, \leq \rangle$ are Classical Mereologies, and each is isomorphic to $\langle F, \leq^F \rangle$.

We may think of the composite of the operations of going “up” from $\langle E, \sqsubseteq \rangle$ to $\langle F, \leq^F \rangle$ and “down” to $\langle H, \leq \rangle$ as a single operation that is applied to $\langle E, \sqsubseteq \rangle$ to yield $\langle H, \leq \rangle$; this is more natural for our application, but harder to define in general. It can be done, however, yielding the generally defined transformation Φ_ρ .⁸

Overview of the construction

The construction of $\langle H, \leq \rangle$ from $\langle D, \mathbf{N}_0 \rangle$ proceeded by five steps.

Given

$$\langle D, \mathbf{N}_0 \rangle$$

take a reflexive and transitive closure:

$$\Phi_r(\Phi_t(\langle D, \mathbf{N}_0 \rangle)) = \langle D, \mathbf{N} \rangle$$

add suitable sets of given objects, along with part-like relations (\in , \subseteq) between them and the given objects and on them:

$$\Phi_1(\langle D, \mathbf{N} \rangle) = \langle E, \mathbf{P}_0 \rangle$$

⁸For a fully general definition, we need some way to tell apart the members of a cluster that are of lower *rank* from the others; in our application, these were members of D rather than of E° . Assuming that our set theory provides a natural way to rank everything in the universe (as does Zermelo-Fraenkel set theory with ur-elements, choice, and foundation) a general transformation Φ_ρ on arbitrary $\langle X, \mathbf{R} \rangle$ may be defined by first applying Φ_t , then, taking a cluster to be a maximal set of members of X that bear \mathbf{R}^t to one another, for each cluster, choosing its single lowest ranked member, if there is one, and the union of all its lowest-ranked sets, otherwise. Φ_ρ is then defined by taking the “chosen” items as carrier set and taking the “inherited” relation.

take a transitive closure:

$$\Phi_t(\langle E, \mathbf{P}_0 \rangle) = \langle E, \mathbf{P} \rangle$$

take the overlap-implication:

$$\Phi_o(\langle E, \mathbf{P} \rangle) = \langle E, \sqsubseteq \rangle$$

and then choose “least or sums” as representatives:

$$\Phi_\rho(\langle E, \sqsubseteq \rangle) = \langle H, \leq \rangle.$$

Each of these steps preserves important aspects of the structures involved, and there are a couple of senses in which the structure of Classical Mereology is a natural “fixed-point” for this sequence of transformations.

From Classical Mereology to itself

Suppose that $\langle D, \mathbf{N}_0 \rangle$ is itself a Complete Classical Mereology (CCM).⁹ Then F is related back to $\langle D, \mathbf{N}_0 \rangle$ as follows. For any $[x], [y] \in F$, with $x, y \in E$, if $x, y \in D$, then $[x] \leq^F [y]$ iff $x \mathbf{N}_0 y$; if $x, y \in E^\circ$, then there is a unique $b \in D$ with $b \in x$, and a unique $c \in D$ with $c \in y$, and $([x] \leq^F [y]$ iff $b \mathbf{N}_0 c$)—in fact, for each other $z \in [x]$, b is the \mathbf{N}_0 -fusion of the members of z , and similarly for c . The map that takes us from $[x]$ to its representative in D (x or b) as in the last sentence is our ρ . In fact, we have

Theorem 5 (*Variation 1*) *If $\langle D, \mathbf{N}_0 \rangle$ is a CCM, then $\langle F, \leq^F \rangle$ is isomorphic to it. More precisely: let the transformation Ψ_1 be*

$$\Phi_a \circ \Phi_o \circ \Phi_t \circ \Phi_1 \circ \Phi_t \circ \Phi_r.$$

Then if \mathfrak{B} is a CCM, $\Psi_1(\mathfrak{B})$ is isomorphic with \mathfrak{B} .

(Variation 2): If $\langle D, \mathbf{N}_0 \rangle$ is a CCM, then $\langle H, \leq \rangle$ is identical with it. More precisely: let Ψ_2 be

$$\Phi_\rho \circ \Phi_o \circ \Phi_t \circ \Phi_1 \circ \Phi_t \circ \Phi_r.$$

⁹A structure is a Complete Classical Mereology if it satisfies any standard set of axioms for Classical Mereology *with the fusion axiom given set-theoretically*. That is, the fusion axiom is a single axiom given with the use of set-theory, rather than an axiom scheme; see section 1.2 of [4].

Then if \mathfrak{B} is a CCM, $\Psi_2(\mathfrak{B}) = \mathfrak{B}$.

To prove this, the main key is Lemma 20 below. Before turning to the proof, observe that, given the above analysis, for each $b \in D$: if b is a Mereological *atom* in $\langle D, \mathbf{N}_0 \rangle$ (i.e., there is no $c \in D$ with $c \neq b$ and $c \mathbf{N}_0 b$) then $[b] = \{b\}$ and $\sigma([b]) = \rho([b]) = b$. Otherwise, $\rho([b]) = b$ and $\sigma([b])$ is the set of b 's \mathbf{N}_0 parts.

Now, of the transformations that we used along the way, three of them involve changing the relation only, and do not alter the carrying set: taking the reflexive or transitive closure (Φ_t and Φ_r), and taking the “overlap inclusion” (Φ_o). These transformations do not alter any structure that is a CCM. This is obvious for Φ_r and Φ_t , since a CCM is already reflexive and transitive. For Φ_o we may use the following Lemma. (The object-language version of this Lemma is called the “strong supplementation” theorem (or, as it may be, axiom) in Classical Mereology).

Lemma 18 *If $\langle X, R \rangle$ is a CCM, then $(\forall x, y \in X)$, if $(\forall z \in X)(z \mathbf{R} x \rightarrow z \circ_{\mathbf{R}} y)$ then $x \mathbf{R} y$.*

We now prove Theorem 5. Let $\langle D, \mathbf{N}_0 \rangle$ be a CCM, and let $\langle E, \sqsubseteq \rangle$ arise from it as described above, by applying $\Phi_o \circ \Phi_t \circ \Phi_1 \circ \Phi_t \circ \Phi_r$.

Lemma 19 $(\forall b, c \in D) (b \sqsubseteq c \leftrightarrow b \mathbf{P} c \leftrightarrow b \mathbf{N}_0 c)$.

Proof: Clearly, $\mathbf{P} \upharpoonright D$ is just \mathbf{N}_0 , so we need only show that the step from \mathbf{P} to \sqsubseteq does not add anything: $\sqsubseteq \upharpoonright D$ is the same relation. We get this from Lemma 18.

Last, we need a lemma telling us that for each $i \in E^\circ$, that there is a unique “small representative” $b \in D$; b is the \mathbf{N}_0 -fusion of i . It is a theorem of CCM that if for each non-empty subset i of the domain, there is a unique fusion of it in this sense: a thing $f(i)$ such that for all y , y overlaps $f(i)$ if and only if it overlaps a member of i . Given i in E° , let $f(i) \in D$ be its $\langle D, \mathbf{N}_0 \rangle$ -fusion.

Lemma 20 $(\forall i \in E^\circ) (\forall b \in D) ((b \sqsubseteq i \wedge i \sqsubseteq b) \leftrightarrow b = f(i))$

Proof: That $f(i) \sqsubseteq i$ is clear from the fusion properties of $f(i)$; that $i \sqsubseteq f(i)$ is clear from those properties and Lemmas 1 and 4. Uniqueness follows

basically from those properties with Lemmas 5, 18, and 19, and the anti-symmetry of \mathbf{N}_0 .

This suffices to show Theorem 5.

Final reflections

We also note a couple results that help to show under what conditions our constructions “leave intact” the structure of \mathbf{N}_0 . Consider the “Strong Supplementation” axiom of Classical Mereology as applied to \mathbf{N} :

$$(\forall x, y \in D)((\forall b \in D)(b \mathbf{N} x \rightarrow b \circ_{\mathbf{N}} y) \rightarrow x \mathbf{N} y)$$

One result is that this holds if and only if $\mathbf{N} = \sqsubseteq \upharpoonright D$. A further easy result is that \mathbf{N} is anti-symmetric iff \mathbf{P} is. Moreover, \mathbf{N} is anti-symmetric iff there are no “proper cycles” (in D) under \mathbf{N}_0 , where a proper cycle is a finite sequence a_1, \dots, a_n with $n > 2$, with $a_1 = a_n$, and for each $i \leq n$, $a_i \neq a_{i+1}$ and $a_i \mathbf{N}_0 a_{i+1}$.

Further, if $\langle D, \mathbf{N}_0 \rangle$ is structurally “well-behaved” in that it features no proper cycles and the resulting $\langle D, \mathbf{N} \rangle$ obeys Strong Supplementation, then $\mathbf{N} = \leq \upharpoonright D$, since for no $x \in D$ will there be a $y \in D$ such that $[x] = [y]$. Thus, if the naturalistic philosopher’s original part-whole structure is “well-behaved” in this sense, our composite transformation Ψ_2 does fairly little, if any, “damage” to the relation \mathbf{N}_0 over its original domain: the restriction of \leq to that domain is just the transitive and reflexive closure of \mathbf{N}_0 .

So if the original part-whole structure is so “well-behaved” that its relation \mathbf{N}_0 is also already reflexive and transitive (hence identical to \mathbf{N}), then the restriction of \leq to the original domain D is identical with the original relation \mathbf{N}_0 : Ψ_2 has then done nothing but “filled in the gaps,” with objects and relational links, so as to provide mereological fusions for arbitrary subsets of the domain, without adding to, or subtracting from, the original links, on the original objects.

Even if the original \mathbf{N}_0 is not already reflexive and transitive, it may be that \mathbf{N}_0 can be recovered from \mathbf{N} in an interesting way. For example, if \mathbf{N}_0 is irreflexive, but transitive, then \mathbf{N}_0 is just \mathbf{N} but with all self-links removed. And even if \mathbf{N}_0 is not transitive, it might still be formally “well-behaved” in this sense: for all $x, y \in D$, $x \mathbf{N}_0 y$ iff $(x \mathbf{N} y$ and there is no $z \in D$ such

that $x \mathbf{N} z$ and $z \mathbf{N} y$); i.e., a part in the most basic sense is an immediate part in the transitive closure of the most basic sense. It is natural to think that this condition might hold in the non-classical mereological systems considered by Koslicki in [6] and in Fine [1]. The system(s) considered in Fine [2], or some important sub-class of them, might also satisfy this condition; the notion of *component* in [2] might be taken as a candidate for our \mathbf{N}_0 .

These remarks should give a taste for the sort of refinements of the results we might reach by further exploration of the kind of technique explored in this paper. A broad statement of the general idea is that if the naturalistic mereologist's part-whole relation on its given domain obeys some apparently very weak formal constraints, it will be possible to define out of it, assuming constructions with set theory, a closely related structure which obeys much more stringent formal constraints that might be favored by the formalistic philosopher, such as those of Classical Mereology, in such a way that the original structure can be recovered as a sub-structure. In this way, the naturalistic mereologist might make peace with the formalistic one, provided the formalistic one is prepared to grant a special status (e.g., being *natural*, or *carving at the joints*, to use a metaphor favored by Sider in [9]) belonging uniquely to that particular sub-structure—to its objects and part-whole relation. Or, put another way, their original dispute might turn out to be merely verbal, the two simply using the words “part” and “object” in different, but ultimately mutually recognizable, ways.

Coda: quick response to some concerns about sets

As we discussed briefly above, the project will only succeed if sets, or some replacement for sets, are granted the sorts of properties the formalistic philosopher ascribes to typical objects, e.g., being located. There are three points about this feature of our project that we may briefly address in closing.

First, it might be thought that if we grant sets location, then we will have a great many co-located sets, e.g., the set d whose members are all the dogs (and nothing else), the set $\{d\}$, the set $\{d, \{d\}\}$, and so forth. If d inherits location from its members, why shouldn't these other sets? Call two sets “materially equivalent” if the transitive closure of the one's member-

ship is identical with the transitive closure of the other's. The reply to this concern would begin by suggesting that if two distinct sets are materially equivalent, then they are *qualitatively* indiscernible: they have the same basic physical properties. The next step would be to argue that it is acceptable for many purposes to pretend that qualitatively indiscernible sets are identical. The expectation would then be that the output of our Ψ_2 transformation captures exactly the right level of distinction among sets: two sets that are materially equivalent with the same element of the output of Ψ_2 are not, for many purposes, different; and every set is materially equivalent with a unique set-or-object in the output of Ψ_2 .

Second, as an alternative to arguing for treating materially equivalent sets as the same (in some contexts), we could find a replacement for sets throughout the entire construction of Ψ_2 . Interestingly, we could use plural quantification over the originally given domain, so that, for example, the role played by a doubleton $\{x, y\} \in E^\circ$ (with $x, y \in D$ and $x \neq y$) would now be played by those things such that: x is one of them, y is one of them, and nothing else is one of them. Arguably, there should be even less resistance to treating pluralities as having properties like location, and there is no problem about there being "too many of them" constructible out of the basic, given, objects.

Third, there is a concern that, given that sets do not change their members over time, they remain unsuited to play the roles of objects. There is much to say about this concern, and here we can only note that it seems worth exploring the possibility that considerations about time will only complicate the story, but not fundamentally change it. For example, if it can be agreed by both the naturalistic and formalistic philosopher that parthood may adequately be treated as a three-placed relation, so that we say " x is part of y at time t " instead of the bare " x is part of y ," then we should consider how all of our formalization might be re-cast accordingly. Or perhaps we may take objects to have temporal parts, or, more non-traditionally, take (some) sets to change their members.¹⁰

¹⁰See [3] for a discussion of the interaction of formalistic mereology with time and tense. The main idea pursued in [3] is to re-conceive formalistic mereology while taking tense (or metaphysical modality) seriously, and allowing objects (including fusions) to change their parts. To wed, in a natural way, the approach in [3] with the idea in this paper would seem to require a set theory in which sets can change their members. Such a set theory should be buildable by modifying untensed set theory in something like the

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manner that [3] modifies untensed Classical Mereology to yield a tensed mereology.

Here is the barest sketch of how this would go. Naïve Set Theory consists of the axiom of Extensionality (sets x and y are the same iff x and y have the same members) together with the Naïve Comprehension scheme for set existence. The scheme is this (for any predicate $\phi(x)$ in which x occurs free and y does not, an instance of the scheme is): there exists at least one set y such that: for all x , $x \in y$ iff $\phi(x)$. Let Tensed Naïve Set Theory be Tensed Extensionality (sets x and y are the same iff *it is always the case that* x and y have the same members) together with a tensed correlate of Comprehension: there exists at least one set y such that *it is always the case that* for all x ($x \in y$ iff $\phi(x)$). An instance of this scheme thus implies that there is a set y such that at every time, for every x , $x \in y$ at that time iff x is (at that time) a dog. This set would have no members when there are no dogs, and its membership would wax and wane with the existence of dogs. Of course the tensed scheme inherits the inconsistency of the Naïve Comprehension scheme; to find a reasonable, consistent tensed set theory, one would modify ZFC with ur-elements, or the like.