§1 Classical Mereology

Primitive Ideology: Two-placed relation $\leq$, first-order logic with identity.

Definitions:

x overlaps y: $x \circ y$ defined as $\exists z(z \leq x \wedge z \leq y)$

b fuses the property ϕ*:* $Fu(b, [x|\phi_x])$ defined as
$\forall x(\phi_x \rightarrow x \leq b) \wedge \forall y(y \leq b \rightarrow \exists x(\phi_x \wedge y \circ x))$

Axioms: (universal closures of)

$\leq$ is reflexive: $x \leq x$
$\leq$ is transitive: $x \leq y \wedge y \leq z \rightarrow x \leq z$
Strong Supplementation: $\forall z(z \circ x \rightarrow z \circ y) \rightarrow x \leq y$
Fusion Existence: $\exists x \phi_x \rightarrow \exists b\, Fu(b, [x|\phi_x])$
Anti-symmetry: $x \leq y \wedge y \leq x \rightarrow x = y$

Theorem

Fusion uniqueness: $Fu(b, [x|\phi_x]) \wedge Fu(c, [x|\phi_x]) \rightarrow b = c$

§2 Four operations on relations

Let R be any relation and let $f(R)$ be the "field" of R, the things that either bear R to something or something bears R to. Let E be any things. Define

$\rho_E(R)$ as the reflexive closure of R with respect to E:
$\rho_E(R)(x, y)$ iff either $R(x, y)$ or ($x = y$ and $x, y \in E$).
$\tau(R)$ as the transitive closure, or "ancestral," of R:
$\tau(R)(x, y)$ iff there is some sequence $a_1, \ldots, a_n$ of things such that for every $i < n$, $R(a_i, a_{i+1})$ and $a_1 = x$ and $a_n = y$.
$\circ_R$ as the relation of (left-)R-overlap:
$x \circ_R y$ iff $\exists z(R(z, x) \wedge R(z, y))$
$\sigma(R)$ as the "overlap-closure" of R:
$\sigma(R)(x, y)$ iff $(\forall b \in f(R))(b \circ_R x \rightarrow b \circ_R y)$.

§3 Some nice behavior

For any R and E:

(3.1) $\rho_E(R)$ is reflexive, and $\tau(R)$ is transitive, and R is a "sub-relation" of each;
(3.2) if R is reflexive on E, then $\rho_E(R) = R$;
(3.3) if R is transitive, then $\tau(R) = R$;
(3.4) (hence, no matter what R is):
$\rho_E(\rho_E(R)) = \rho_E(R)$,
$\tau(\tau(R)) = \tau(R)$,
(3.5) $\rho_E(\tau(R)) = \tau(\rho_E(R))$;
(3.6) if R is reflexive and transitive, then
R is a sub-relation of $\sigma(R)$, and $\sigma(R)$ is transitive and is reflexive (on its field),
$\sigma(\sigma(R)) = \sigma(R)$,
$\circ_R = \circ_{\sigma(R)}$;
(3.7) (and, no matter what R is):
if $S = \sigma(\tau(\rho_E(R)))$, then $\sigma(S) = \tau(S) = \rho_E(S) = S$.

§4 Connections to mereological axioms

Let R be any relation, and E be any things. We have already noted that $\tau(\rho_E(R))$ obeys the axioms of reflexivity and transitivity. Less obvious but provable is that $\sigma(\tau(\rho_E(R)))$ obeys also the Strong supplementation axiom.

§5 Combining relations: example

Let R be any relation whose field is a subset of some chosen set D (none of whose members is a non-well-founded set), and let E be **the set of all the non-empty, non-singleton subsets of** D. Let F be the union of D and E, and let T be the relation on F such that

$$T(x,y) \leftrightarrow (R(x,y) \vee x \in y).$$

Let P be $\sigma(\tau(\rho_F(T)))$. Then:

(5.1) P (on F) satisfies the first four axioms of Classical Mereology listed above, including the Fusion-existence axiom.

Moreover, the subset relation (on E) is a sub-relation of the restriction of P to E. (Under some conditions, e.g., if R were empty, then the restriction of P to E would be identical with the subset relation on E.)

(5.2) If R (on D) already satisfied the first three axioms itself, then the restriction of P to D is identical with R.

(5.3) if R on D satisfied the first four axioms then the structure of P on F is "quasi-isomorphic" to the structure of R on D: isomorphic, when violations of Anti-symmetry are factored out.

This last result means that if we "iterate" by applying, to P on F, the process that took us from the relation R (on domain D) to the relation P (on F), the result is quasi-isomorphic again.

§6 Combining relations: in general

Suppose we have two relations, R and S, whose fields are subsets of D and E respectively, where D and E are disjoint. Let Q be any relation such that for any x, y, if $Q(x,y)$ then ($x \in D$ and $y \in E$). Let T be the union of R, S, and Q (i.e. $T(x,y) \leftrightarrow (R(x,y) \vee S(x,y) \vee Q(x,y))$). Let F be the union of D and E.

Now we consider the relationship between $\tau(\rho_D(R))$ and $\tau(\rho_F(T))$; we find that $\tau(\rho_D(R))$ is exactly what you get if you restrict $\tau(\rho_F(T))$ to D. Thus, the "expansion" of $\langle D, \tau(\rho_D(R)\rangle$ to $\langle F, \tau(\rho_F(T))\rangle$ "does not disturb anything" within D.

Similarly, $\sigma(\tau(\rho_D(R)))$ is exactly what you get if you restrict $\sigma(\tau(\rho_F(T)))$ to D.

§7 Finean mereology

Operationalism: Take various compositional operations as more basic than notions of part-whole. Define *component* in terms of the operations, and define *part* in terms of component.

Examples:

Mereological sum (fusion) of Socrates and Plato is
$\sum_m$(Socrates, Plato) (= $\sum_m$(Plato, Socrates), = $\sum_m$(Socrates, $\sum_m$(Socrates, Plato)));
The set { Socrates, { Socrates, Plato }} is
$\sum_\in$(Socrates, $\sum_\in$(Socrates, Plato)) ($\neq \sum_\in$(Socrates, Plato));
Sequence of Socrates (first) and Plato (second) is
$\sum_{seq}$(Socrates, Plato) ($\neq \sum_{seq}$(Plato, Socrates)).

Principles the holding or failing to hold of which helps characterize operations: (here the equations are required to be "regular" (same objects on both sides))

Absorption: $\sum(\ldots, x, x, \ldots, \ldots, y, y, \ldots, \ldots) = \sum(\ldots, x, \ldots, y, \ldots)$;
Collapse: $\sum(x) = x$;
Leveling: $\sum(\ldots, \sum(x, y, z, \ldots), \ldots, \sum(u, v, w, \ldots) \ldots) = \sum(\ldots, x, y, z, \ldots, \ldots, u, v, w, \ldots, \ldots)$;
Permutation: $\sum(x, y, z, \ldots) = \sum(y, z, x, \ldots)$ (and similarly for other permutations).

Component and part:

Let $\sum_k$ be some composition operation.

Define *x is a k-component of y* as "y can be immediately reached by applying $\sum_k$ to some things-in-a-sequence that include x."
Formally (?): $\exists \ldots_1 \exists \ldots_2\ y = \sum(\ldots_1, x, \ldots_2)$.
Fine writes: "x is a component of y if y is the result of applying $\sum$ to x or to x and some other objects. In other words, y should be of the form $\sum(x_1, x_2, \ldots)$ where at least one of $x_1, x_2, \ldots$ is x."

Define *x is a k-part of y* as "y can be reached from x through a sequence of k-components."
I.e., k-part is τ(k-component), i.e., the ancestral.
Define *x is a K-part of y* as the ancestral of the relation *being **some** kind of component of*.

An objection to operationalism

The novel apparatus of things-in-a-sequence quantification already involves at least one significantly part-like notion. Thus, not all cases of parthood are to be explained on the above model. E.g., to say

y is a sequence of which Socrates is a component

is to say

$\exists \ldots_1$ (Socrates **is one*** **of** $\ldots_1$ and $y = \sum_{seq}(\ldots_1)$).

The objection is that, however it is made out, the **is one*** **of** relation is part-like (at least as part-like as the relevant relation between Socrates and the sequence $\langle$Socrates, Plato$\rangle$). (Yes, it is "cross-categorial," like the "is one of" of more familiar plural logic, but so what?)

Also, these things-in-a-sequence items seem an awful lot like sequences. This is not to say that Fine is forced to identify $\sum_{seq}$(Socrates, Plato) with $\sum_{seq}$($\langle$Socrates, Plato$\rangle$); the latter is a one-entry-long sequence whose one entry is the two-entry-long sequence that is the former. But the structure of the ideology of sequences and their components may be indistinguishable from that of the ideology of the … "things" and the "**ones***" of them.