Connected Graph Partitions and Minimal Free Resolutions of Toppling Ideals

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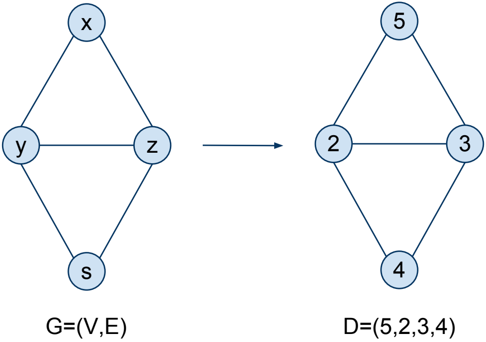
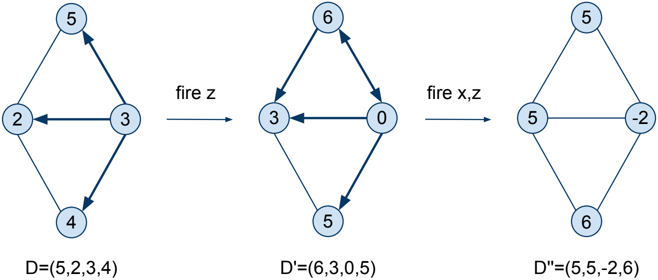
Divisors on a Graph

A **divisor** on a graph G is a formal product $D = \sum_{v \in V_G} d_v v$ where each $d_v \in \mathbb{Z}$. We denote the set of divisors on G by $\mathrm{Div}(G)$.



Vertex and Set Firing

From a divisor we may **fire** a vertex or a set of vertices to obtain a new divisor.



Effective Divisors

A divisor D is **effective with respect to** a set $P \subseteq V_G$ if the divisor obtained by firing P from D is nonnegative.

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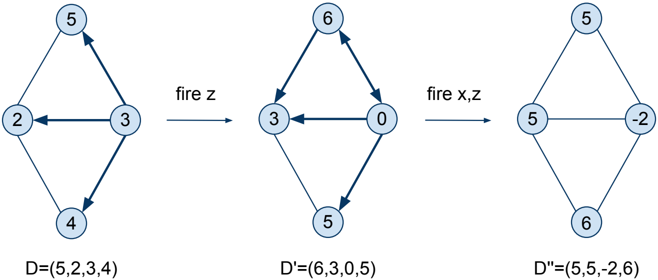

Connected Partitions and Their Minimal Effective Divisors

Question

Let $V_1, V_2, \cdots, V_k$ be a connected k-partition of G. When is a divisor effective with respect to the sequence of set firings $V_1, V_2, \cdots, V_k$? Is there a minimum one among these divisors?

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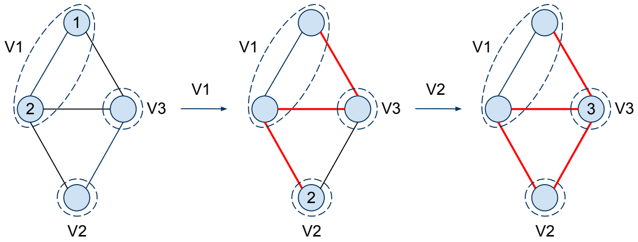
Answer:
Yes, there is unique minimal divisor effective to $V_1, V_2, \cdots, V_k$.

Minimal Effective Divisors: An Example



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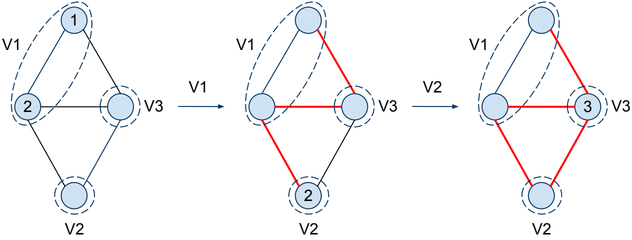
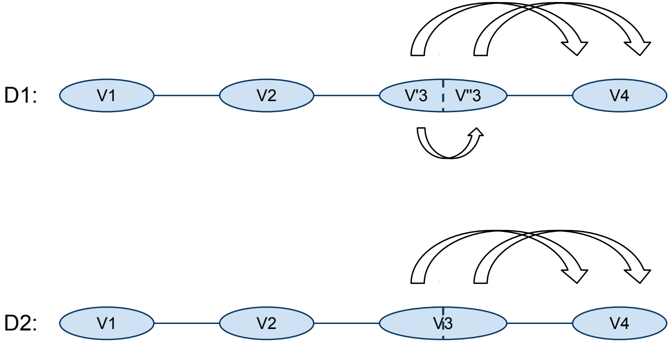


Theorem

Let $v \in V_i$ $(1 \leq i \leq k)$, *then*

$$D_{\min}(v) = \sum_{i<j\leq k} \mathrm{wt}(v, V_j)$$

Minimal Effective Divisors: An Observation



$$D_1 - D_2 = \sum_{v \in V_3'} \mathrm{wt}(v, V_3'')v$$

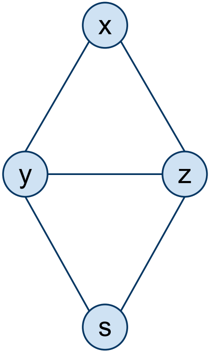
The Laplacian Matrix

Definition

Let $G = (V, E)$ be a graph and $v_1, v_2, \cdots, v_n$ be an ordering of V, the **Laplacian matrix** of G, denoted by Δ_G, is defined by

$$\Delta_G = D_G - A_G$$

where D_G, A_G are the degree and adjacency matrices of G, resp.



x y z s

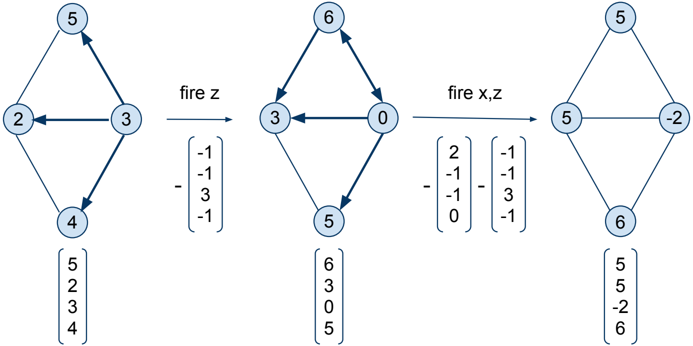
$$\Delta_G = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

x
y
z
s

The Laplacian Encodes Firing Instructions

Observation

The column vector in Δ_G correponding to a vertex $v \in V_G$ gives the difference in divisors caused by firing v.



Divisors in Monomials

Assumptions

For now on, let $G = (V, E)$ *be a graph with* $V = \{v_1, v_2, \cdots, v_n\}$*, and let* $R = \mathbb{C}[x_1, x_2, \cdots, x_n]$*. For a divisor* $D \geq 0$ *on* G*, we define*

$$x^D = \prod_{v \in V} x_v^{D_v}$$

The Toppling Ideal

Definition

The **toppling ideal** of G, denoted by I_G, is defined by

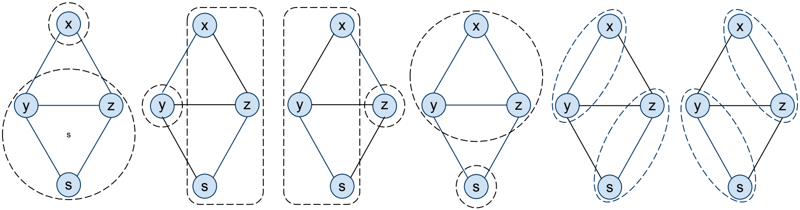
$$I_G = \mathrm{Span}_{\mathbb{C}}\{x^D - x^E : D, E \in \mathrm{Div}(G)_{\geq 0}, D - E \in \mathrm{Image}(\Delta_G)\}$$

Example:

If $\Delta_G = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$, then $x_1^2 - x_2x_3, x_2^3 - x_1x_3x_4 \in I_G$.

Generators of Toppling Ideal

It is known that the generators of toppling ideal can be obtained from connected 2-partitions of the graph...



$$I_G = (x^2 - yz, y^3 - xzs, z^3 - xys, s^2 - yz, xy^2 - z^2s, xz^2 - y^2s)$$

Notation

Let $g_1, g_2, \cdots, g_m$ be a set generators of I_G, we define M_1 to be the $1 \times m$ matrix with the g_i's as its entries so that $I_G = Image(M_1)$.

Relations Among the Generators: Second Syzygies

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Question

Let $g_1, g_2, \cdots, g_m$ be the generators of I_G. What are the relations among these generators? (That is, for what $f_1, f_2, \cdots, f_m \in R$ do we have $f_1 g_1 + f_2 g_2 + \cdots f_m g_m = 0$? Or, what is the kernel of M_1?)

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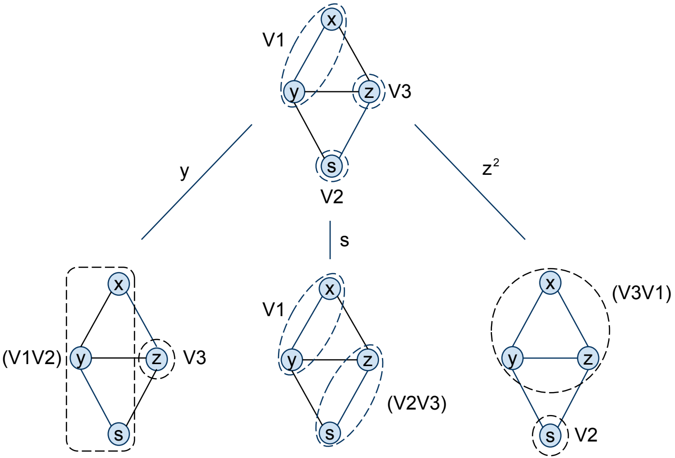
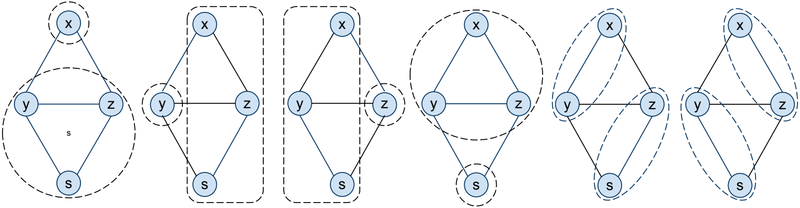
The answer comes from the connected 3-partitions of G.

A Relation from a 3-Partition



$$y(xys - z^3) + s(sz^2 - xy^2) + z^2(zy - s^2)$$
$$= xy^2s - yz^3 + s^2z^2 - xy^2s + z^3y - s^2z^2 \quad = 0$$

Recall the ordering of the generators of I_G:



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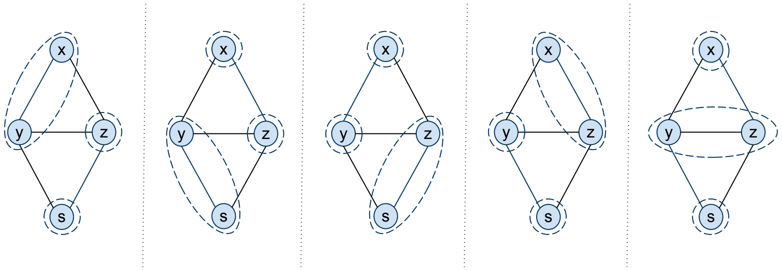
Second Syzygies from 3-Partitions



$$M_2 = \begin{bmatrix} 0 & 0 & -ys & -z^2 & -y^2 & -zs & 0 & 0 & s^2 - yz \\ 0 & 0 & 0 & 0 & -z & -x & -s & -z & 0 \\ -y & -s & -x & -y & 0 & 0 & 0 & 0 & 0 \\ -z^2 & -xy & 0 & 0 & 0 & 0 & -xz & -y^2 & yz - x^2 \\ -s & -z & 0 & 0 & x & y & 0 & 0 & 0 \\ 0 & 0 & z & x & 0 & 0 & -y & -s & 0 \end{bmatrix}$$

Remarks

- We saw that each column in M_2 gives a relation among the generators of I, hence $\mathrm{Image}(M_2) \subseteq \mathrm{Kernel}(M_1)$.

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- We investigated the relations of a finite set of generators, which themselves have a finite set of generators.
- We may go on to find the relations of this new set of generators, i.e., we may go on to find M_3 with $\mathrm{Image}(M_3) = \mathrm{Kernel}(M_2)$. This is the idea of a resolution.

Free Resolutions

Definition

Let R be a ring and M be an R-module, a **free resolution** of M is an exact sequence of free R-modules

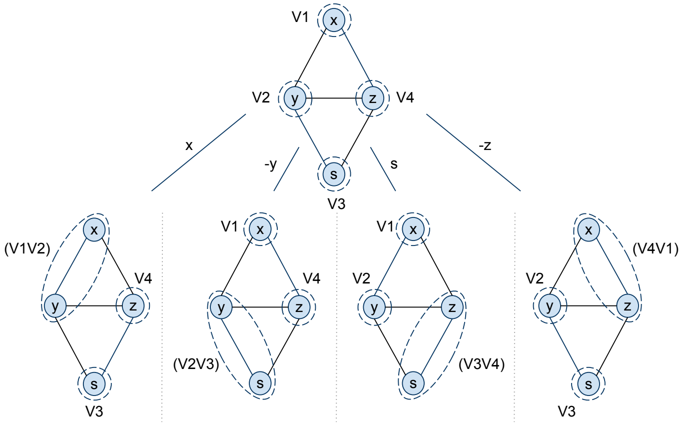
$$\mathcal{F} : \ F_0 \xleftarrow{\varphi_1} F_1 \longleftarrow \cdots \xleftarrow{\varphi_r} F_r \longleftarrow \cdots$$

such that $\mathrm{Coker}(\varphi_1) = M$. The image of φ_i is called the i-th **syzygy module** of M.

Note: We are interested in $M := R/I_G = \mathrm{Coker}(\varphi_1)$.

Third Syzygies from 4-Partitions

We showed how an ordered 3-partition gives a second syzygy.
The 4-partitions give third syzygies similarly.



Third Syzygies from 4-Partitions

$$M_3 = \begin{bmatrix} 0 & x & -y & 0 \\ x & 0 & 0 & -y \\ -s & -y & 0 & 0 \\ 0 & 0 & y & s \\ z & s & 0 & 0 \\ 0 & 0 & -s & -z \\ 0 & -z & x & 0 \\ -z & 0 & 0 & x \\ -y & 0 & -z & 0 \end{bmatrix}$$

Minimal Free Resolution of the Toppling Ideal

Conjecture

A minimal free resolution of the toppling ideal associated with a graph may be derived from connected partitions of the graph. In particular, a minimal set of generators of the $(i-1)$-th syzygy of this ideal may be obtained from connected i-partitions of the graph.

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A minimal free resolution of R/I_G in our example graph is given by

$$\mathcal{F}: \; R \xleftarrow{\varphi_1} R^6 \xleftarrow{\varphi_2} R^9 \xleftarrow{\varphi_3} R^4 \longleftarrow 0$$

where the matrix representation of φ_i is M_i for each $1 \leq i \leq 3$.

Remarks On the Conjecture

- For families of lattice ideals recognizable as toppling ideals, our conjecture may easily produce their minimal free resolutions.

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- The definition of a free resolution does not assume it is finite. When M is finitely generated, Hilbert Syzygy Theorem guarantees so, as is also expected from our conjecture.
- We didn't discuss the "connected" and "minimal" in the title. In fact, this minimality comes from this connectedness.