Combination Locks and Permutations

An Exploration Through Analysis

Tim Sasaki

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Introduction

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- How many combinations are possible?
  - If we push buttons one at a time, then the number of combinations is simply $n!$ since we are not repeating buttons.
  - However, what if we are allowed to push multiple buttons at once?

Introduction

An example for $n = 4$

Possible combinations are:

- $\{1,2\},\{3\},\{4\}$
- $\{2,3,4\},\{1\}$
- $\{1,3\},\{2,4\}$
- $\{1\},\{2\},\{3\},\{4\}$

In particular, the set of multiple-button-push combinations contains the set of single-button-push combinations, so we know there are at least $n!$ combinations (the only instance when the number of combinations is actually equal to $n!$ is for $n \in \{0,1\}$).

Introduction

Exhaustive checking works fine for small n.

Combinations for $n = 3$

$(\{1\},\{2\},\{3\})$, $(\{1\},\{3\},\{2\})$, $(\{2\},\{1\},\{3\})$, $(\{2\},\{3\},\{1\})$, $(\{3\},\{1\},\{2\})$, $(\{3\},\{2\},\{1\})$, $(\{1,2\},\{3\})$, $(\{1,3\},\{2\})$, $(\{2,3\},\{1\})$, $(\{1\},\{2,3\})$, $(\{2\},\{1,3\})$, $(\{3\},\{1,2\})$, $(\{1,2,3\})$

There are 13 possible combinations for $n = 3$.

However, the number of combinations as n increases becomes very large.

- We know that there are at least $n!$ combinations, and the factorials blow up quickly.

Introduction

For $n = 6$, this gives us at least $6! = 720$ combinations.

Then, we are motivated to try and derive a formula to compute the number of combinations for a lock with n buttons, since writing out each of 720 combinations is not desirable.

Introduction

Let a_n be the number of combinations for an n-button lock. Then, note that a_0, the number of combinations for a 0-button lock, is equal to 1 (the only combination is no buttons pressed).

Further, for any $n > 0$, a valid combination comprises an initial push of k buttons, for $1 \le k \le n$, followed by a combination formed with the remaining $n - k$ buttons.

Introduction

This leads us to a recurrence relation:

$$a_n = \sum_{k=1}^{n} \binom{n}{k} a_{n-k} = \binom{n}{1} a_{n-1} + \binom{n}{2} a_{n-2} + \cdots + \binom{n}{n-1} a_1 + \binom{n}{n} a_0,$$

where $\binom{n}{k}$ represents the number of ways to pick the initial k buttons and a_{n-k} represents the number of ways to choose all of the remaining $n-k$ buttons.

Introduction

Let's try out our formula using $n = 3$. Using our recurrence relation,

$$\begin{aligned} a_3 &= \binom{3}{1} a_2 + \binom{3}{2} a_1 + \binom{3}{3} a_0 \\ &= \binom{3}{1}(3) + \binom{3}{2}(1) + \binom{3}{3}(1) \\ &= (3 \cdot 3) + (3 \cdot 1) + (1 \cdot 1) \\ &= 13 \end{aligned}$$

This matches the value we had earlier. Naturally, we would like to try and find a closed-form equation instead of a recursive one.

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This is a task we will accomplish through the use of some infinite series and integrals.

For the duration of the talk, we will adopt the convention that $0^0 = 1$ strictly for notational convenience.

A new recursive formula

We now take our relation

$$a_n = \sum_{k=1}^{n} \binom{n}{k} a_{n-k} = \binom{n}{1} a_{n-1} + \binom{n}{2} a_{n-2} + \cdots + \binom{n}{n-1} a_1 + \binom{n}{n} a_0,$$

and put in the values of the binomial coefficients:

$$\begin{aligned} a_n &= \frac{n!}{(n-1)!1!} a_{n-1} + \frac{n!}{(n-2)!2!} a_{n-2} + \cdots + \frac{n!}{1!(n-1)!} a_1 + \frac{n!}{0!n!} a_0 \\ &= n! \left(\frac{a_{n-1}}{(n-1)!1!} + \frac{a_{n-2}}{(n-2)!2!} + \cdots + \frac{a_1}{1!(n-1)!} + \frac{a_0}{0!n!} \right). \end{aligned}$$

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We notice a relationship in the terms of the sum between each a_k and the first factorial in the denominator. We thus define a new term $b_n = a_n / n!$ and rewrite the relation:

$$a_n = n! \left(\frac{b_{n-1}}{1!} + \frac{b_{n-2}}{2!} + \cdots + \frac{b_0}{n!} \right)$$

A new recursive formula

$$a_n = n!\left(\frac{b_{n-1}}{1!} + \frac{b_{n-2}}{2!} + \cdots + \frac{b_0}{n!}\right)$$

Finally, since $b_n = a_n \,/\, n!$, we divide by $n!$ to get

$$b_n = \frac{a_n}{n!} = \left(\frac{b_{n-1}}{1!} + \frac{b_{n-2}}{2!} + \cdots + \frac{b_0}{n!}\right) = \sum_{k=1}^{n} \frac{b_{n-k}}{k!}.$$

This relation is a bit simpler than our original one, so it may be easier to find a closed equation for b_n first and then get one for a_n.

The calculus begins

Theorem 1

For all $n > 0$,

$$\frac{1}{2(\ln 2)^n} \le b_n \le \frac{1}{(\ln 2)^n}$$

We consider the generating function

$$f(x) = \sum_{n=0}^{\infty} b_n x^n.$$

It can be shown using Theorem 1 that the series is absolutely convergent for $|x| < \ln 2$.

And continues.

Then, for $|x| < \ln 2$,

$$
\begin{aligned}
f(x) &= b_0 + \sum_{n=1}^{\infty} b_n x^n \\
&= 1 + \left(\frac{b_0}{1!}x^1\right) + \left(\frac{b_1}{1!}x^2 + \frac{b_0}{2!}x^2\right) + \left(\frac{b_2}{1!}x^3 + \frac{b_1}{2!}x^3 + \frac{b_0}{3!}x^3\right) + \cdots \\
&= 1 + \left(\frac{b_0}{1!}x^1 + \frac{b_1}{1!}x^2 + \frac{b_2}{1!}x^3 + \cdots\right) + \left(\frac{b_0}{2!}x^2 + \frac{b_1}{2!}x^3 + \cdots\right) + \cdots \\
&= 1 + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{b_{n-k}}{k!}x^n \\
&= 1 + \sum_{k=1}^{\infty}\frac{x^k}{k!}\sum_{n=0}^{\infty} b_n x^n \\
&= 1 + (e^x - 1) f(x)
\end{aligned}
$$

From this, we get that $f(x) = \frac{1}{2-e^x}$ for $|x| < \ln 2$.

And continues..

If $\sum_{n=0}^{\infty} b_n x^n$ is the power series representation of f at 0, then we know that the coefficients, b_n, have the form

$$b_n = \frac{f^{(n)}(0)}{n!}.$$

Recall that $b_n = a_n \,/\, n!$, which implies that

$$a_n = f^{(n)}(0),$$

or

$$a_n = \frac{d^n}{dx^n}\left(\frac{1}{2-e^x}\right)\bigg|_{x=0}.$$

However, this formula is still problematic, since we must calculate each intermediary derivative to get to the n^{th} one if we wish to use it.

And continues...

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as long as the series converges, which is for $|e^x/2| < 1$.

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$$\frac{e^x}{2} < 1 \implies e^x < 2 \implies x < \ln 2.$$

Then, because $0 < \ln 2$,

$$a_n = \frac{d^n}{dx^n}\left[\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{e^x}{2}\right)^k\right]\Bigg|_{x=0} = \frac{1}{2}\sum_{k=0}^{\infty}k^n\left(\frac{e^0}{2}\right)^k = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k}.$$

And now, integrals

We can approximate the sum

$$\sum_{k=0}^{\infty} \frac{k^n}{2^k} \approx \int_0^{\infty} \frac{x^n}{2^x}\, dx,$$

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$$\int_0^{\infty} \frac{x^n}{2^x}\,dx = \int_0^{\infty} \frac{(u/(\ln 2))^n}{2^{u/(\ln 2)}} \cdot \frac{1}{\ln 2}\,du = \frac{1}{(\ln 2)^{n+1}} \int_0^{\infty} \frac{u^n}{\left(2^{(1/\ln 2)}\right)^u}\,du$$

$$= \frac{1}{(\ln 2)^{n+1}} \int_0^{\infty} \frac{u^n}{e^u}\,du.$$

Integrals, continued

We focus now on the new integral and use integration by parts n times (choosing the constituent parts wisely):

$$\int_0^\infty \frac{u^n}{e^u}\,du = \lim_{b\to\infty} \sum_{k=0}^{n} -\frac{n!}{(n-k)!}\frac{u^{n-k}}{e^u}\Bigg|_0^b = \sum_{k=0}^{n} \frac{n!}{(n-k)!}\frac{0^{n-k}}{e^0}.$$

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We note that every term in the sum is equal to 0 except the last, which is

$$\frac{n!}{(n-n)!}\frac{0^0}{e^0} = \frac{n!}{0!}\frac{1}{1} = n!.$$

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Referring back to our estimation of a_n, this gives us

$$a_n = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k} \approx \frac{1}{2}\int_0^\infty \frac{x^n}{2^x}\,dx = \frac{1}{2(\ln 2)^{n+1}}\int_0^\infty \frac{u^n}{e^u}\,du = \frac{n!}{2(\ln 2)^{n+1}}.$$

A brief recap

We've derived the following formulas for a_n so far:

1. $$a_n = \sum_{k=1}^{n} \binom{n}{k} a_{n-k} = \binom{n}{1} a_{n-1} + \binom{n}{2} a_{n-2} + \cdots + \binom{n}{n-1} a_1 + \binom{n}{n} a_0$$

2. $$a_n = \frac{d^n}{dx^n}\left(\frac{1}{2-e^x}\right)\Bigg|_{x=0}$$

3. $$a_n = \frac{1}{2}\sum_{k=0}^{\infty} \frac{k^n}{2^k}$$

4. $$a_n \approx \frac{n!}{2(\ln 2)^{n+1}}$$

We now wish to determine just how accurate our last approximation is.

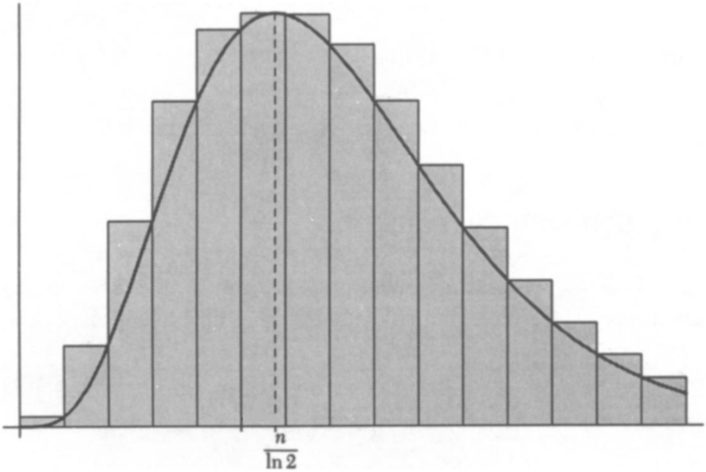
Some more calculus

We begin by defining a function g such that for $n \in \mathbb{N}$, for all $x \in [0, \infty)$, $g(x) := x^n / e^x$. The First Derivative Test will show us that g is increasing on $[0, n/\ln 2]$ and decreasing on $[n/\ln 2, \infty)$. Hence, g has a local maximum at $x = n/\ln 2$.

We are going to use rectangles to approximate the integral $\displaystyle\int_0^\infty \frac{x^n}{e^x}\,dx$. Instead of using the height at left-or-right endpoints, however, we will be using the suprema (least upper bound) and infima (greatest lower bound) of g in each subinterval. These are called *Darboux sums.*

A pictoral interlude

Break the interval $[0,\infty)$ into subintervals $[0,1],[1,2],\cdots$, and define $j = \lfloor n/\ln 2 \rfloor$.



Back to work

Recall that g has a max at $x = n/\ln 2$. The value of g at this point is

$$\frac{(n/\ln 2)^n}{2^{n/\ln 2}} = \frac{n^n}{\left(2^{1/\ln 2}\right)^n (\ln 2)^n} = \left(\frac{n}{e \ln 2}\right)^n.$$

Then, the sum of the areas of the rectangles from the previous slide is

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Then, the sum of the areas of the rectangles from the previous slide is

$$\sum_{k=1}^{j} \frac{k^n}{2^k} + \left(\frac{n}{e \ln 2}\right)^n + \sum_{k=j+1}^{\infty} \frac{(k)^n}{2^k}$$

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$$\sum_{k=1}^{j} \frac{k^n}{2^k} + \left(\frac{n}{e\ln 2}\right)^n + \sum_{k=j+1}^{\infty} \frac{(k)^n}{2^k}$$
$$= \sum_{k=0}^{\infty} \frac{k^n}{2^k} + \left(\frac{n}{e\ln 2}\right)^n = 2a_n + \left(\frac{n}{e\ln 2}\right)^n.$$

An answer at last

The approximation is clearly greater than the value of the integral, so this gives us

$$\int_0^\infty \frac{x^n}{2^x}\,dx < 2a_n + \left(\frac{n}{e\ln 2}\right)^n \implies -\frac{1}{2}\left(\frac{n}{e\ln 2}\right)^n < a_n - \frac{n!}{2(\ln 2)^{n+1}}.$$

A similar process for an underestimation of the integral yields

$$a_n - \frac{n!}{2(\ln 2)^{n+1}} < \frac{1}{2}\left(\frac{n}{e\ln 2}\right)^n.$$

Hence, we have

$$\left|a_n - \frac{n!}{2(\ln 2)^{n+1}}\right| < \frac{1}{2}\left(\frac{n}{e\ln 2}\right)^n,$$

which means our approximation is within $\frac{1}{2}\left(\frac{n}{e\ln 2}\right)^n$ of a_n.

An unexpected result

The error bound we found is a function that increases with n. Despite this, we can show that

$$\lim_{n\to\infty} \frac{a_n}{n!/\left[2(\ln 2)^{n+1}\right]} = 1.$$

The proof uses our derived error bounds, Stirling's Formula, and the Squeeze Theorem to show that

$$\lim_{n\to\infty} \left| \frac{a_n}{n!/\left[2(\ln 2)^{n+1}\right]} - 1 \right| = 0.$$

An unexpected result

This may initially seem counter-intuitive, since the error is increasing, but consider as a simple example $n^3 + n^2$ and n^3. The error between them is n^2, which increases quadratically with n, but

$$\lim_{n \to \infty} \frac{n^3 + n^2}{n^3} = 1.$$

Finally, some combinatorics

Instead of approximating a_n via the methods we have gone through, we can also simply just evaluate the infinite sum. For $n \geq 0$, define

$$h_n(x) = \sum_{k=0}^{\infty} k^n x^k.$$

Recalling our formula

$$a_n = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k^n}{2^k},$$

it is clear that

$$a_n = \frac{1}{2} h_n\left(\frac{1}{2}\right).$$

So, by finding formulas for each h_n, we can find a formula for a_n.

But first, some preliminary calculus (Ha!)

Looking at $h_0(x)$, we have

$$h_0(x) = \sum_{k=0}^{\infty} k^0 x^k = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad -1 < x < 1.$$

We now look at the differentiation of the series:

$$h_n'(x) = \sum_{k=0}^{\infty} k^n k \cdot x^{k-1} = \sum_{k=0}^{\infty} k^{n+1} x^{k-1},$$

from which we can get

$$xh_n'(x) = \sum_{k=0}^{\infty} k^{n+1} x^k = h_{n+1}(x).$$

Yet another recurrence

Given our newly found recurrence, $xh'_n(x) = h_{n+1}(x)$, we look at its particular values for $n \leq 5$:

$$h_0(x) = \frac{1}{1-x}$$

$$h_1(x) = \frac{x}{(1-x)^2}$$

$$h_2(x) = \frac{x+x^2}{(1-x)^3}$$

$$h_3(x) = \frac{x+4x^2+x^3}{(1-x)^4}$$

$$h_4(x) = \frac{x+11x^2+11x^3+x^4}{(1-x)^5}$$

$$h_5(x) = \frac{x+26x^2+66x^3+26x^4+x^5}{(1-x)^6}$$

Some patterns of note

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1. The numerator contains a polynomial of degree n.
2. The coefficients of the polynomial appear to be symmetric.
3. The denominator is always $(1-x)^{n+1}$.

We define now $A_{n,k}$ to be the coefficient of x^k in h_n. For example, given

$$h_4(x) = \frac{x + 11x^2 + 11x^3 + x^4}{(1-x)^5},$$

$A_{4,3} = 11$.

This is the last recurrence relation

Theorem 2

For all $n \geq 1$,

$$h_n(x) = \frac{\sum_{k=1}^{n} A_{n,k} x^k}{(1-x)^{n+1}}$$

where $A_{n,k}$ is defined by the following recurrence:

$$A_{n,1} = A_{n,n} = 1, \qquad A_{n+1} = kA_{n,k} + (n+2-k)A_{n,k-1}, \qquad \text{for } 2 \leq k \leq n$$

The proof of Theorem 2 follows by induction.

Since we have a formula for $A_{n,k}$ now, we will compute them for the first few values of n.

Like Pascal's Triangle, only not

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 4 & & 1 & & \\ & 1 & & 11 & & 11 & & 1 & \\ 1 & & 26 & & 66 & & 26 & & 1 \end{array}$$

These do in fact match the coefficients we found earlier. Recall (or learn anew) that the numbers in the n^{th} row of Pascal's Triangle sum to 2^n. Let's sum the rows of this triangle to see if we get anything interesting:

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				1				
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$$1 + 1 = 2$$

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$$1 + 4 + 1 = 6$$

$$1 + 11 + 11 + 1 = 24$$

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$$
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 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 4 & & 1 & & \\
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1 & & 26 & & 66 & & 26 & & 1
\end{array}
$$

These do in fact match the coefficients we found earlier. Recall (or learn anew) that the numbers in the n^{th} row of Pascal's Triangle sum to 2^n. Let's sum the rows of this triangle to see if we get anything interesting:

$$1 = 1$$

$$1 + 1 = 2$$

$$1 + 4 + 1 = 6$$

$$1 + 11 + 11 + 1 = 24$$

$$1 + 26 + 66 + 26 + 1 = 120$$

Like Pascal's Triangle, only not

				1				
			1		1			
		1		4		1		
	1		11		11		1	
1		26		66		26		1

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- These are the factorials!

Eulerean numbers

The $A_{n,k}$'s we defined earlier are, in fact, already well known in combinatorics as the Eulerean numbers, and they denote the number of n-permutations with k increasing runs. As a reminder, an n-permutation is a permutation of $1, 2, \cdots, n$. A definition of an increasing run is as follows:

Increasing run

For an n-permutation $(s_1\ s_2 \cdots s_n)$, a sequence of terms $s_i\ s_{i+1} \cdots s_k$ where $i \leq k$, $s_j < s_{j+1}$ for $i \leq j < k$, and either $k = n$ or $s_k > s_{k+1}$.

Example: The permutation (1 4 3 5 2) has the increasing runs (1 4), (3 5), and (2).

No calculus appears on this slide

The patterns we noticed earlier are in fact properties of the Eulerean numbers:

Theorem 3

For all $n \geq 1$,

(a) $\sum_{k=1}^{n} A_{n,k} = n!$,

(b) $A_{n,k} = A_{n,n+1-k}$ for $1 \leq k \leq n$.

Part (a) is obvious from the definition of $A_{n,k}$. A proof of part (b) appears in [*Advanced Combinatorics* by L. Comtet, p. 242].

Recalling that $a_n = (1/2)\,h_n(1/2)$, we can now say

$$a_n = \left(\frac{1}{2}\right) \frac{\sum_{k=1}^{n} A_{n,k}(1/2)^k}{(1-(1/2))^{n+1}} = \sum_{k=1}^{n} A_{n,k} 2^{n-k}.$$

Interpreting the formula

The equation $a_n = \sum_{k=1}^{n} A_{n,k} 2^{n-k}$ has a natural interpretation for the lock combinations. If we take an n-permutation with k increasing runs, we can delimit the runs as seen for (4 1 3 2 5 6):

$$(4\,|\,1\,3\,|\,2\,5\,6).$$

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This gives us 2^{n-k} combinations for each n-permutation with k increasing runs, of which there are $A_{n,k}$.

A formula does exist for $A_{n,k}$ (see Comtet, pp. 243), we can combine it with the above formula to get

$$a_n = \sum_{k=1}^{n} \sum_{i=0}^{k-1} (-1)^i \binom{n+1}{i} (k-i)^n \, 2^{n-k}.$$

The Stirling numbers of the second kind

Stirling number of the second kind

For $n \in \mathbb{N}$ and $1 \leq k \leq n$, the Stirling number of the second kind, denoted $S(n,k)$ is equal to the number of ways to partition the set $\{1, 2, \cdots, n\}$ into k unordered non-empty subsets.

A combination lock is of course an *ordered* partition, and if we have k subsets, there are $k!$ ways to order them. Hence,

$$a_n = \sum_{k=1}^{n} k! S(n,k).$$

A formula for $S(n,k)$ exists as well (Comtet, pp. 204-205):

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^n \binom{k}{i} (k-i)^n.$$

Thus, we get our last formula for a_n:

$$a_n = \sum_{k=1}^{n} \sum_{i=0}^{k-1} (-1)^n \binom{k}{i} (k-i)^n.$$

Wrapping up

Here are all of our formulas for a_n:

1. $a_n = \sum_{k=1}^{n} \binom{n}{k} a_{n-k} = \binom{n}{1} a_{n-1} + \binom{n}{2} a_{n-2} + \cdots + \binom{n}{n-1} a_1 + \binom{n}{n} a_0$
2. $a_n = \frac{d^n}{dx^n} \left(\frac{1}{2 - e^x} \right) \Big|_{x=0}$
3. $a_n = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k^n}{2^k}$
4. $a_n \approx \frac{n!}{2(\ln 2)^{n+1}}$
5. $a_n = \sum_{k=1}^{n} \sum_{i=0}^{k-1} (-1)^i \binom{n+1}{i} (k-i)^n \, 2^{n-k}$
6. $a_n = \sum_{k=1}^{n} \sum_{i=0}^{k-1} (-1)^n \binom{k}{i} (k-i)^n$

References

1. L. Comtet, *Advanced Combinatorics*, D. Reidel, Dordrecht-Holland, 1974

2. D. Velleman and G. Call, *Permutations and combination locks, Mathematics Magazine* **68** (1995) 243-252.