Symmetric Configurations in the Abelian Sandpile Model

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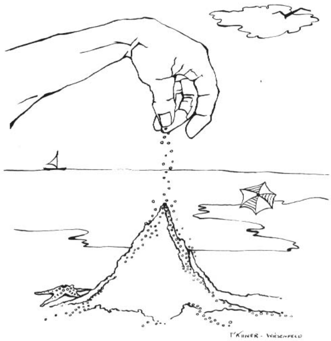
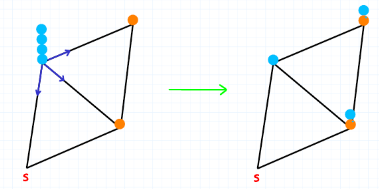
Self-organized criticality

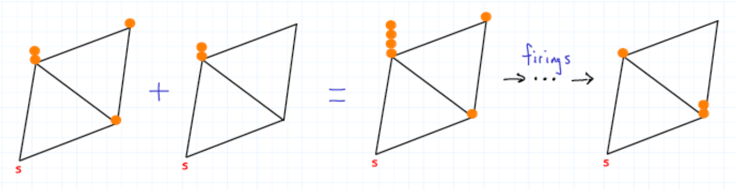
Bak, Tang and Wiesenfeld invented the idea of "self-organized criticality", also observed that builing a sandpile by dropping grains on sand, one obtains a cone of sand.



- connection between the recurrent states of the sandpile model and the dimer model
- connection between the dimer model and viral tiling theory

Abelian Sandpile Model



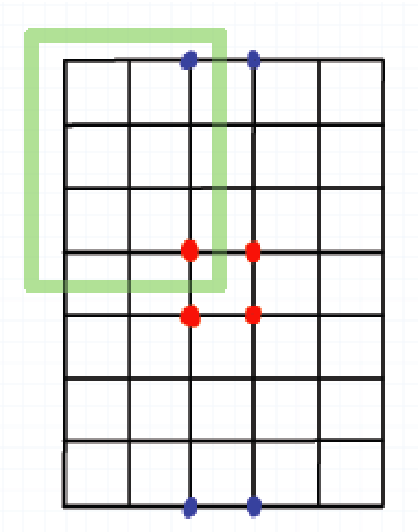


Definition

A configuration is recurrent if it is stable and given any configuration a, there exists a configuration b, s.t. $a + b = c$.

The recurrent configurations form the sandpile group.

Symmetric Sandpiles



Reduced Laplacian

Definition

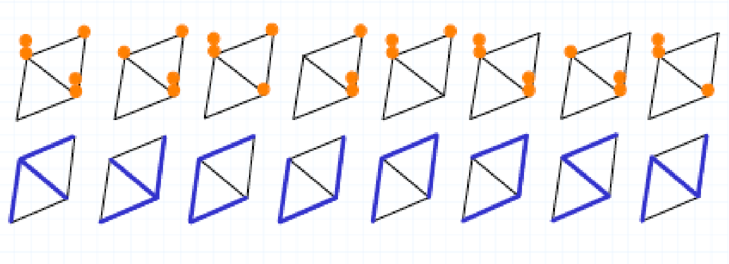
The reduced Laplacian of a graph encodes the adjacency matrix and its firings. It is called reduced because we remove the row and column for the sink vertex.

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 3 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

Matrix Tree Theorem, KPW

Theorem (Matrix-Tree Theorem)

The number of spanning trees is the determinant of the reduced Laplacian.

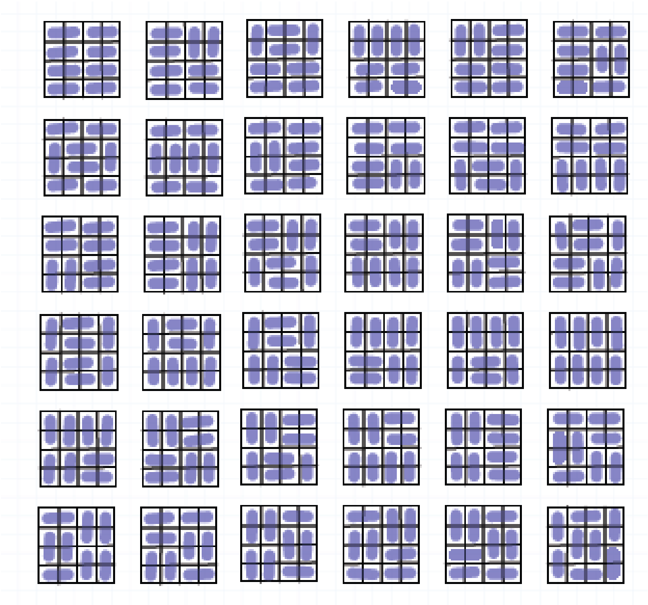


Theorem (Kenyon,Propp, Wilson)

Trees on a planar graph are in bijection with perfect matchings on a corresponding graph.

- Furthermore, perfect matchings $\iff$ domino tilings.

Tilings



Grid graphs without symmetry

Theorem

Let $H_{2m,2n+1}$ *denote the tilings on a* $2m \times 2n+1$ *grid graph. Let* $NG_{m,n}$ *denote the number of recurrents of the* $m \times n$ *grid graph with the right corner elements firing* 1 *to the sink and the right edge elements, except for the corner elements, firing* 0 *to the sink.*
Then, $|H_{2m,2n+1}| = NG_{m,n}$*. In addition, these are counted by the following Chebyshev polynomial formula:* $\displaystyle\prod_{k=1}^{n/2} U_m\left(i\cos\left(\frac{k\pi}{n+1}\right)\right)$.

Chebyshev Polynomials

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

$$U_0(x) = 1$$
$$U_1(x) = 2x$$
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

$$T_n(x) = \det \begin{pmatrix} x & 1 & & & \\ 1 & 2x & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 2x \end{pmatrix}.$$

$$U_n(x) = \det \begin{pmatrix} 2x & 1 & & & \\ 1 & 2x & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 2x \end{pmatrix}.$$

The roots of T_n are $x_k = \cos \frac{(2k-1)\pi}{2n}$, $k = 1, \cdots, n$.

Similarly, the roots of U_n are $x_k = \cos \frac{k\pi}{n+1}$, $k = 1, \cdots, n$.

Grid graphs without symmetry

The reduced laplacian of this grid graph with special firings is

$$\triangle = \begin{bmatrix} A_n & -I_n & & & \cdots & & 0 \\ -I_n & A_n & -I_n & & & & \\ & \ddots & \ddots & \ddots & & & \vdots \\ & & -I_n & A_n & -I_n & & \\ \vdots & & & \ddots & \ddots & \ddots & \\ & & & & -I_n & A_n & -I_n \\ 0 & & \cdots & & & -I_n & A_n \end{bmatrix}$$

where I_n is the $n \times n$ identity matrix and where the A_n matrices are of the following form:

$$\begin{bmatrix} 4 & -1 & & & \cdots & & 0 \\ -1 & 4 & -1 & & & & \\ & \ddots & \ddots & \ddots & & & \vdots \\ & & & & -1 & 4 & -1 \\ 0 & & \cdots & & & -1 & 3 \end{bmatrix},$$

Grid graphs without symmetry

$$\begin{aligned} T_{11} &= -A_n S_{m-1} + S'_{m-1} \\ &= -A_n S_{m-1} + S_{m-2} \\ &= U_m\left(\frac{A_n}{2}\right) \end{aligned}$$

Grid graphs without symmetry

$$T_{11} = -p(A) = -\prod_{k=0}^{m-1}\left(A_n - 2\cos\frac{k\pi}{n+1}I_n\right).$$

$$\det\triangle = (-1)^n \det\left(-\prod_{k=0}^{m-1}\left(A_n - 2\cos\frac{k\pi}{n+1}I_n\right)\right)$$
$$= \prod_{k=0}^{m-1}\det\left(A_n - 2\cos\frac{k\pi}{n+1}I_n\right).$$

Letting χ_n denote the characteristic polynomial of A_n and setting $t_{m,k} := 2\cos\frac{k\pi}{n+1}$, the above result rewrites as

$$\boxed{\det\triangle = \prod_{k=0}^{m-1}\chi_n(t_{m,k})}. \qquad (1)$$

Grid graphs without symmetry

$$N_{2m,2n+1} = 2^{2mn} \prod_{j=1}^{2m} \prod_{k=1}^{2n+1} \left| \cos^2 \frac{j\pi}{2m+1} + \cos^2 \frac{k\pi}{2n+2} \right| =$$

$$\left| \prod_{j=1}^{2m} \prod_{k=1}^{2n+1} \left(-2\cos \frac{j\pi}{2m+1} - 2i \cos \frac{k\pi}{2n+2} \right) \right|^{1/2}$$

$$\begin{aligned} x_k &= 2i \cos \frac{k\pi}{2n+2} \\ s_{2m,j} &= \prod_{j=1}^{2m} \left(x_k - 2\cos \frac{j\pi}{2m+1} \right), \end{aligned}$$

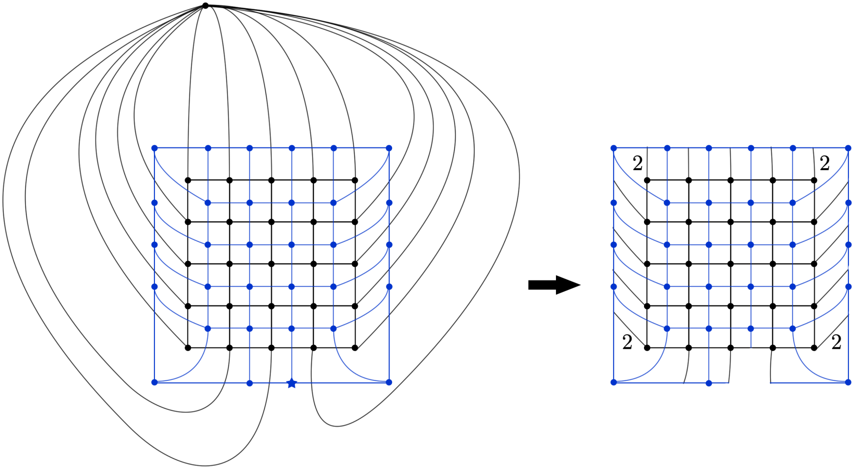
$$s_{2m,j} = \prod_{j=1}^{2m} \left(x_k - 2\cos \frac{j\pi}{2m+1} \right) = U_{2m}\left(\frac{x_k}{2} \right),$$

Grid graphs without symmetry

Theorem

Let $G_{m,n}$ *be the number of recurrents on a* $m \times n$ *grid graph and let* $H_{2m+1,2n+1}$, *for* $n > 1$, *be the number of tilings on a* $2m+1 \times 2n+1$ *grid graph with a cell removed.*
Then, $G_{m,n} = H_{2m+1,2n+1}$.

Grid graphs without symmetry



Grid graphs with Klein Four Group symmetry

Theorem

The following are equal:

(i) *The number of domino tilings of a $2m \times 2n$ grid*

(ii) *the number of symmetric recurrents on a $2m \times 2n$ grid*

(iii) $\displaystyle\prod_{k=1}^{n} U_{2m}\left(i\cos\left(\frac{k\pi}{2n+1}\right)\right)$

(iv) $\displaystyle\prod_{h=1}^{2m}\prod_{k=1}^{2n}\left(4\cos^2\frac{h\pi}{2m+1}+4\cos^2\frac{k\pi}{2n+1}\right)$ *from Kasteleyn*

Summary

- det symmetric Laplacian $=$ $\#$ symmetric recurrents
- symmetric Laplacian $=$ reduced Laplacian of a related graph. The det symmetric laplacian $=$ $\#$ trees
- trees - perfect matchings

Grid graphs with Klein Four Group symmetry

Theorem

The following are equal:

(i) *The number of domino tilings of a 2m × 2n Möbius grid*

(ii) *the number of symmetric recurrents on a 2m × 2n-1 grid*

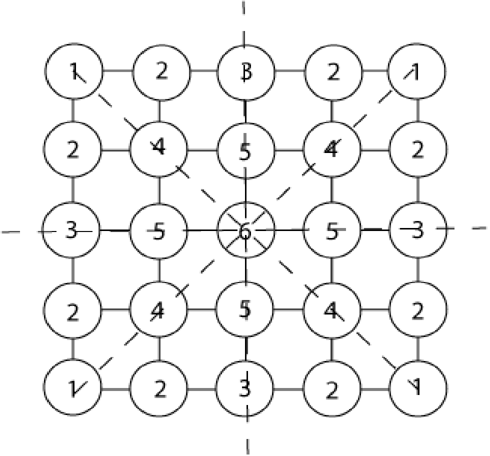
(iii) $$2^m \prod_{k=1}^{m} T_{2n}\left(1 + 2\cos\left(\frac{k\pi}{2m+1}\right)\right)$$

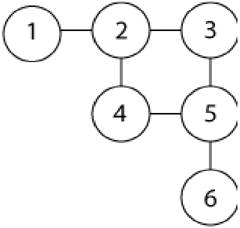
(iv) $$\prod_{h=1}^{m}\prod_{k=1}^{n}\left(4\cos^2\frac{h\pi}{2m+1} + 4\sin^2\frac{(4k-1)\pi}{4n}\right)$$ *from Lu+Wu*

(v) *the number of weighted domino tilings of a 2m × 2n grid, as shown below:*

Grid graphs with Klein Four Group symmetry

Grid graphs with dihedral symmetry

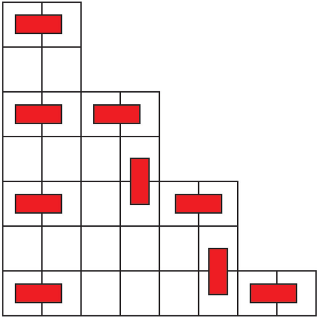




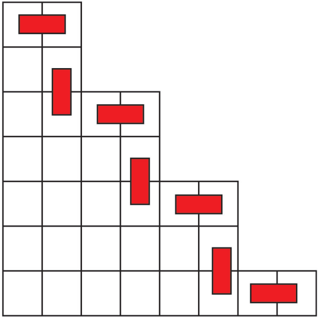
Grid graphs with dihedral symmetry

Theorem

Let $\mathcal{H}_{n-1}$ denote the weighted tilings on the Pachter graph H_{n-1} as shown below for the cases when n is odd or even. Let $NG_{2m,2n}$ denote the number of recurrents of the $2m \times 2n$ grid graph with dihedral symmetry. Then, $\mathcal{H}_{n-1} = NG_{2m,2n}$.



n odd



n even

Grid graphs with dihedral symmetry

Laplacian of the grid graph:

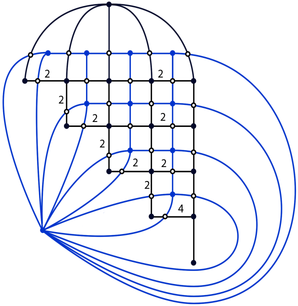
$$\begin{bmatrix} A_{n-5} & -I_{n-6} & & & \cdots & & 0 \\ -I'_{n-6} & A_{n-6} & -I_{n-7} & & & & \\ & \ddots & \ddots & \ddots & & & \vdots \\ & & & & -I'_2 & A_2 & -I_1 \\ 0 & & \cdots & & & -I'_1 & 2 \end{bmatrix},$$

where the A_n matrices are of the following form:

$$\begin{bmatrix} 4 & -2 & & & \cdots & & 0 \\ -1 & 4 & -1 & & & & \\ & \ddots & \ddots & \ddots & & & \vdots \\ & & & & -1 & 4 & -1 \\ 0 & & \cdots & & & -1 & 3 \end{bmatrix},$$

Grid graphs with dihedral symmetry

Next, we create the graph Γ from the transpose of the Laplacian matrix. This graph, and its dual, look like this:



Now, by KPW, we can associate to each symmetric recurrent a weighted domino tiling as in the statement of the proof.

Current work

- bijection between weighted domino tilings on the grid graph and domino tilings on the Mobius strip