Problem 4.10 (Engel)

Part A. Are the eigenfunctions of $\hat{H}$ for the particle in a one-dimensional box also eigenfunctions of the momentum operator $\hat{p}_x$?

Solution

Strategy. The eigenfunctions of $\hat{H}$ are $\psi(x) = \sqrt{\frac{2}{a}} \sin \left( \frac{n \pi x}{a} \right)$. If these are also eigenfunctions of $\hat{p}_x$, they will satisfy $\hat{p}_x \psi = \text{const} \ast \psi$, so let's evaluate the left side of this equation and see what happens.

Execution.

$\hat{p}_x = -i \hbar \frac{\partial}{\partial x}$

$\hat{p}_x \psi = -i \hbar \frac{\partial \psi}{\partial x}$

The derivative can be evaluated as follows:

$$
\partial_x \left( \sqrt{\frac{2}{a}} \sin \left( \frac{n \pi x}{a} \right) \right) \\
\sqrt{2} \left( \frac{1}{a} \right)^{3/2} n \pi \cos \left( \frac{n \pi x}{a} \right)
$$

The result is a cosine function, which is obviously not a constant multiple of the original sine function. $\psi$ is not an eigenfunction of $\hat{p}_x$.

Part B. Calculate the average value of $p_x$ for the case where $n = 3$ and 5, i.e., $\psi(x) = \sqrt{\frac{2}{a}} \sin \left( \frac{3 \pi x}{a} \right)$ or $\sqrt{\frac{2}{a}} \sin \left( \frac{5 \pi x}{a} \right)$

Solution

Strategy. The average value of $p_x$ is also called the expectation value, $<p_x>$, and is obtained as the value of the following integral:

$$
<p_x> = \int_0^a \psi \hat{p}_x \psi \, dx
$$

Note: this integral relies on these facts: 1) $\psi$ is a real function, and 2) $\psi$ has been normalized.

Execution.
The average momentum of the particle for both the $n = 3$ and $n = 5$ states is zero. You may have noticed that both of the expressions that needed to be integrated contained $\cos*\sin$ and was an odd function, so the integrals were guaranteed to vanish.

**Comment.** The text would like us to generalize this result to the following: the average momentum of the particle in a box is zero regardless of $n$. You might be reluctant to make this generalization, however, because the two examples that were chosen both involved odd values of $n$. To guarantee the result, we calculate the average value for an arbitrary case of $n = m$.
\[ \psi = \sqrt{\frac{2}{a}} \sin\left(\frac{m \pi x}{a}\right) \]

\[ \sqrt{2} \sqrt{\frac{1}{a}} \sin\left(\frac{m \pi x}{a}\right) \]

\[ p\psi = (-i \hbar) \frac{\partial}{\partial x} \psi \]

\[ -i \sqrt{2} \left(\frac{1}{a}\right)^{3/2} \hbar m \pi \cos\left(\frac{m \pi x}{a}\right) \]

\[ \psi^* \psi = \psi \psi^* \]

\[ -2 i \hbar m \pi \cos\left(\frac{m \pi x}{a}\right) \sin\left(\frac{m \pi x}{a}\right) \]

\[ \int_0^a \psi^* \psi \, dx \]

\[ -i \hbar \sin[m \pi]^2 \frac{1}{a} \]

\[ \text{Simplify}[\% , m \in \text{Integers}] \]

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