Problem 2.11 (Lowe)

Prove the following statement: Any linear combination of degenerate eigenfunctions of $\hat{H}$ is also an eigenfunction of $\hat{H}$.

Proof

Suppose we have two eigenfunctions of $\hat{H}$ with the same eigenvalue $\lambda$. In other words, there exists:

$$\hat{H} \psi = \lambda \psi \text{ and } \hat{H} \phi = \lambda \phi$$

Now define a third function that is a linear combination of the first two:

$$\chi = a \psi + b \phi$$

We want to prove:

$$\hat{H} \chi = \lambda \chi$$

Let's start by expanding the left side of this equation:

$$\hat{H} \chi = \hat{H} (a \psi + b \phi)$$

Quantum mechanical operators are linear, so

$$\hat{H} \chi = \hat{H} (a \psi + b \phi) = (a \hat{H} \psi + b \hat{H} \phi)$$

$$= a \lambda \psi + b \lambda \phi = \lambda (a \psi + b \phi) = \lambda \chi$$