Problem 2.30 (Engel)

Show that the rotation operator does not change the length of an arbitrary vector.

rotation operator = \[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]

arbitrary vector = \[
\left( \begin{array}{c}
a \\ b
\end{array} \right) = a \left( \begin{array}{c}1 \\ 0 \end{array} \right) + b \left( \begin{array}{c}0 \\ 1 \end{array} \right)
\]

Solution

The arbitrary vector has length \( \sqrt{a^2 + b^2} \). A useful way to see this is to notice that the vector is the sum of two perpendicular vectors of length \( a \) and \( b \), respectively (see diagram), so we can get the sum-vector's length by applying the Pythagorean theorem.

To show that the rotation matrix does not affect the length of \( (a, b) \), I show that 1) rotation produces two new vectors of lengths \( a \) and \( b \), respectively, and 2) the new vectors are perpendicular.

The rotation operator produces the following results on each of the unit vectors:

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}1 \\ 0 \end{pmatrix} = \begin{pmatrix}(1) \cos \theta - (0) \sin \theta \\ (1) \sin \theta + (0) \cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}
\]

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}0 \\ 1 \end{pmatrix} = \begin{pmatrix}(0) \cos \theta - (1) \sin \theta \\ (0) \sin \theta + (1) \cos \theta \end{pmatrix} = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}
\]

The rotated unit vectors are still unit vectors:

length \( \left( \begin{array}{c}
\cos \theta \\ \sin \theta
\end{array} \right) = \sqrt{\cos^2 \theta + \sin^2 \theta} = \sqrt{1} = 1 \)

(the length of the other rotated unit vector is left as an exercise for you).

Thus, the effect of rotation on column vector \( (a, b) \) is:
\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
[\begin{pmatrix}
a \\
0
\end{pmatrix}
+ \begin{pmatrix}
b \\
0
\end{pmatrix}]
\]

= \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
a \\
1
\end{pmatrix}
+ \begin{pmatrix}
b \\
0
\end{pmatrix}

= a \begin{pmatrix}
\cos \theta \\
\sin \theta
\end{pmatrix}
+ b \begin{pmatrix}
-\sin \theta \\
\cos \theta
\end{pmatrix}
\]

So the rotated vector is the sum of two vectors of length \(a\) and \(b\), respectively. The new vectors are perpendicular (orthogonal) because their dot product is zero:

\[
\text{dot product} \ (\cos \theta, \sin \theta).(-\sin \theta, \cos \theta) = -\cos \theta \sin \theta + \sin \theta \cos \theta = 0
\]

This completes the problem.